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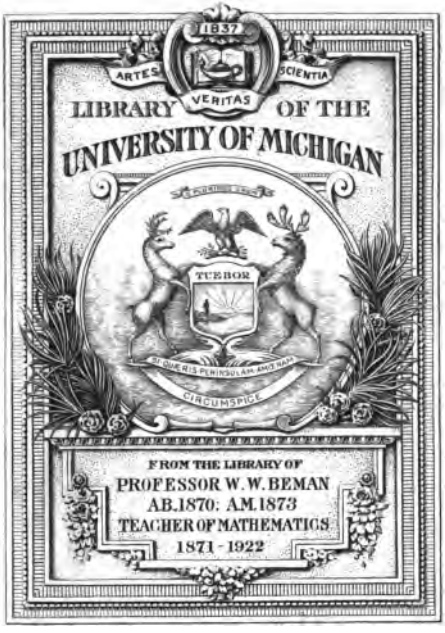

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IN ADDITION TO THOSE

PUBLISHED IN THE "EDUCATIONAL TIMES,"

AND TWO

APPENDICES.

EDITED BY

W. J. C. MILLER, B.A.,

REGISTRAR

OF THE

GENERAL MEDICAL COUNCIL.

VOL. XLVI.

LONDON:

FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

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1453. (Professor Sylvester, F.R.S.)—Show that a given point is *inside*, *outside*, or *neither* inside nor outside, a given conic, according as the product of the discriminant by the point-function is *positive*, *negative*, or *zero*..... 112

2231. (Professor Sylvester, F.R.S.)—If a system of conics touch a given straight line at a given point at which they have all the same curvature, and pass through a second given point, show that, on taking as axis of coordinates the bisectors of the angle between the given line and the line joining the two given points, the locus of the foci of the conics so taken consists of the two circular cubics

$$2by\,(x^2+y^2)-(ax+by)^2=0,\quad 2ax\,(x^2+y^2)-(ax+by)^2=0,$$

where a, b are the x, y coordinates of the second given point. 48

5178. (Professor Sylvester, F.R.S.)—x, y, z are three given rational integral biquadratic functions of t; and two systems of ratios $a : b : c$, $A : B : C$ are to be found such that $\frac{Ax^2+By^2+Cz^2}{ax^2+by^2+cz^2}$ shall be an integral function of t; show that the problem admits in general of three, and never of more than three, solutions. 61

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6317. (Dr. Hopkinson, F.R.S.)—A viscous fluid occupies the space between two infinite parallel planes, whereof one is fixed, whilst the other moves parallel to itself with simple harmonic motion $A \cos nt$. Show that the consequent tangential force on the fixed plane has a maximum value per unit of area $= 2A\lambda\mu n (\cosh 2\lambda l - \cos 2\lambda l)^{-\frac{1}{2}}$, where l is the distance between the planes, $\lambda^2 = \frac{\rho n}{2\mu}$, and the fluid in contact with a plane is assumed to move with the plane. 72

7191. (Professor Minchin, M.A.)—In the general motion of any fluid, if, with reference to any rectangular axes, the components of distortion at any point are (a, b, c, s_1, s_2, s_3), and those of vortical spin are $(\omega_1, \omega_2, \omega_3)$; and if α is the normal acceleration at any point on any closed surface described in the fluid, prove that

$$\int \alpha d\mathrm{S} = \iiint (\theta^2 + \mathrm{D}_t\,\theta + 2\Omega^2 - 2\Sigma)\, dx\, dy\, dz,$$

where θ is the expansion, Ω the resultant vortical spin, Σ the strain invariant $ab + bc + ca - s_1^2 - s_2^2 - s_3^2$, and $\mathrm{D}_t\,\theta$ the total time-rate of increase of θ; the first integral being taken over the surface, and the second throughout its volume. 61

7305. (Professor Malet, F.R.S.)—Prove that the equation of the tangent cone from the point (x', y', z') to the quadric

$$\mathrm{S} \equiv ax^2 + by^2 + cz^2 + 2hxy + 2gxz + 2fyz + 2lx + 2my + 2ns + d = 0,$$

when referred to its axes, becomes $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0$, $\lambda_1, \lambda_2, \lambda_3$ being the roots of the equation

$$\begin{vmatrix} a-\lambda, & h, & g, & \mathrm{U}_1 \\ h, & b-\lambda, & f, & \mathrm{U}_2 \\ g, & f, & c-\lambda, & \mathrm{U}_3 \\ \mathrm{U}_1, & \mathrm{U}_2, & \mathrm{U}_3, & \mathrm{S}' \end{vmatrix} = 0, \text{ where } \begin{aligned} \mathrm{U}_1 &= ax' + hy' + gz', \\ \mathrm{U}_2 &= hx' + by' + fz', \\ \mathrm{U}_3 &= gx' + fy' + cz'. \end{aligned}$$

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7361. (W. J. C. Sharp, M.A.)—In a circular cubic, prove that (1) the directrices, or chords of contact, corresponding to concyclic foci, pass through the centre of the focal circle, which is the point of contact of one of the tangents parallel to the real asymptote; also (2) the four centres lie at the intersections of the cubic and a rectangular hyperbola which has an asymptote in common with the cubic..... 114

7392. (Rev. T. C. Simmons, M.A.)—Having given two homographic pencils in a plane, show how to draw a transversal whose segments intercepted between homologous rays shall be equal, each to each. 107

7432. (Professor Cochez.) — On place n jetons sur les cases d'un échiquier carré de n^2 cases, de telle sorte qu'il n'y ait qu'un seul jeton dans chacune des colonnes ou des lignes de l'échiquier; démontrer que (1) si l'on désigne par S_n le nombre des solutions symétriques par rapport au centre de l'échiquier, on a $S_{2n+1} = S^{2n} = 2 \cdot 4 \cdot 6 \ldots (2n)$; (2) si l'on désigne par u_n le nombre des solutions symétriques par rapport à l'une des diagonales, on a $u^n = u_{n-1} + (n-1)\, u_{n-2}$, et aussi

$$u^n = 1 + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4} + \frac{n(n-1) \ldots (n-5)}{2 \cdot 4 \cdot 6} + \&c.;$$

et (3) traiter la même question pour les solutions symétriques par rapport aux deux diagonales..... 22

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7434. (The Editor.)—Trace the locus of the feet of the normals drawn from a given point on the conics inscribed in a given square, distinguishing the various forms of the curve for different positions of the given point. 79

7703. (W. G. Lax, B.A.)—A, P, B are three points in order in a straight line; on AB, AP, PB as diameters semicircles are described all on the same side of AB. At P a perpendicular PR to AB is erected meeting the semicircle on AB in R; prove that the circles inscribed to the curvilinear figures ARP, BRP have equal radii, and find the value... 26

7840. (Professor Wolstenholme, M.A., Sc.D.)—If θ be any angle between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, prove that (1) for all real values of n

$$\frac{\sin n\theta}{\cos\theta} = n\sin\theta + n(2^2-n^2)\frac{\sin^3\theta}{3!} + n(2^2-n^2)(4^2-n^2)\frac{\sin^5\theta}{5!} + \ldots,$$

$$\frac{\cos n\theta}{\cos\theta} = 1 + (1^2-n^2)\frac{\sin^2\theta}{2!} + (1^2-n^2)(3^2-n^2)\frac{\sin^4\theta}{4!} + \ldots;$$

(2) if $x = \sin\theta$, and $y = \dfrac{\sin n\theta}{\cos\theta}$ or $\dfrac{\cos n\theta}{\cos\theta}$,

$$(1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + (n^2-1)y = 0. \ldots\ldots 45$$

7998. (Professor Haughton, F.R.S.)—Four points on a quartic lie on a line (A); three other points lie on a line (B); three other points lie on a line (C); there are (of course) two other real points, lying on (B) and (C) respectively; prove that, for every possible quartic passing through the above ten points, the line joining the remaining two real points passes through a fixed point which can be constructed. 24

8137. (The Editor.)—If the velocity of a carriage along a road is proportional to the cube of the cosine of the inclination of the road to the horizon, determine (1) the path of quickest ascent from the bottom to the top of a hemispherical hill, and show that it consists of a spherical curve described by a point of a great circle which rolls on a small circle described about the pole with radius $\frac{1}{2}\pi$, together with an arc of a great circle; and show (2) how the discontinuity is introduced into this problem. 27

8185. (Professor Neuberg.)—On donne la base BC d'un triangle ABC, et le sommet A se meut sur une droite donnée BY. Trouver (1) les lieux du centre de gravité et du cercle des neuf points (droites); (2) l'enveloppe de la droite qui joint ces points (conique); (3) les lieux des sommets B′, C′ du triangle A′B′C′, que forment les tangentes en A, B, C au cercle ABC (parallèle à BC et conique); (4) l'enveloppe de B′C′ (parabole de foyer C); (5) l'enveloppe de la symédiane AA′ (conique de foyer C); (6) le lieu décrit par le centre K des symédianes (conique); (7) les lieux décrits par les points de Brocard (cercle et polaire de conique).... 52

8212. (D. Edwardes.)—If SY, S′Y′ are the focal perpendiculars on a tangent through a point O to an ellipse; SZ, S′Z′ the focal perpendiculars on the other tangent through O: prove that (1) YZ, Y′Z′, and the straight line through O perpendicular to the major axis, are concurrent; and (2), if YZ meet OS′ in L, Y′Z′ meet OS in M, the points L, M, S, S′ are concyclic. 44

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8217. (Professor Catalan.)—On donne une suite indéfinie de termes, dont les deux premiers sont $A_0 = 1$, $A_1 = 4$, et dont les autres, à partir de $n = 2$, sont donnés par la loi de récurrence,

$$n^2 A_n - 4(3n^2 - 3n + 1) A_{n-1} + 32(n-1)^2 A_{n-2} = 0;$$

prouver que tous ces termes sont des *nombres entiers.* 51

8220. (Professor Neuberg.)—On donne une courbe (C) et une angle XOY. Une tangente quelconque à (C) rencontre OX en A, OY en B: trouver (1) la tangente au lieu décrit par l'orthocentre H du triangle OAB; (2) la tangente au lieu décrit par le centre I du cercle OAB; (3) la tangente au lieu du centre de gravité G. 85

8226. (R. Knowles, B.A.)—An ordinate from a focus of an ellipse is produced to meet the auxiliary circle in T; tangents TP, TQ meet the ellipse in P and Q; prove that the radius of the circle TPQ : chord PQ $= (1 + 3e^2)^{\frac{1}{2}} : 4e$, e being the eccentricity of the ellipse. 44

8229. (J. Brill, M.A.)—Through a point P tangents PQ and PR are drawn to a parabola, and circles are drawn through the focus touching the parabola at Q and R respectively; prove that the common chord of these circles passes through the centroid of the triangle PQR. 46

8232. (Rev. T. C. Simmons, M.A.)—Three points, P, Q, R, are taken at random within a given triangle, also a point K whose triangular coordinates are a, b, c. Prove that (1) the chance that Q and R fall on opposite sides of PK is equal to

$$2a^2 \int_{\frac{ab}{1-a}}^{1-b} \left\{ 2b - 6ab^2 + (1-4ab)\left(x + \frac{b^2}{x}\right) - a\left(x^2 + \frac{b^4}{x^2}\right) \right\} dx,$$

together with the five similar expressions obtained by the interchanging of the letters a, b, c; (2) hence, or otherwise, when K is situated on the perimeter, the required chance becomes $\frac{1}{3}$; and (3) when K is situated at the centroid, it becomes $\frac{3}{8}\frac{1}{1} + \frac{4}{2}\frac{0}{4}\frac{0}{3} \log 2$ or ·4968 nearly. 83

8251. (Professor Matz, M.A.)—A candidate is examined in three papers, to each of which m marks are assigned as a maximum; show that, if his total on the three papers is $2m$, there are $\frac{1}{2}(m+1)(m+2)$ ways in which this may occur. 106

8260. (Rev. T. R. Terry, M.A.)—If the normal at P to the parabola $y^2 = 4ax$ meet the curve again at Q, and if $PQ^2 : 4SP^2 = SP : SA$, then $PQ = 6a\sqrt{3}$. 47

8269. (Asparagus.)—From a point O on a parabola there are drawn chords OP, OQ normal to the parabola at P, Q; prove that (1) $\cos POQ = 3\cos\theta$, where θ is the angle which the normal at O makes with the axis; and (2) OP × OQ = latus rectum × diameter of curvature at O. 38

8277. (Professor Neuberg.)—Le cercle osculateur en un point M d'un ellipse rencontre cette courbe en M_1; le cercle osculateur en M_1 rencontre l'ellipse en M_2, etc.; (1) La ligne brisée $MM_1, M_2 \ldots$ peut-elle se fermer? (2) Les cordes telles que MM_1, MM_2, $MM_3 \ldots$ enveloppent des projections orthogonales d'épicycloïdes. 64

8280. (Professor Sáradáranjan Rây, M.A.)—If ρ be the length of a normal from (α, β) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, prove that

$$\frac{\rho^2}{k^2} = \frac{\alpha^2}{(k+a^2)^2} + \frac{\beta^2}{(k+b^2)^2}, \text{ where } k \text{ is given by } \frac{a^2\alpha^2}{(k+a^2)^3} + \frac{b^2\beta^2}{(k+b^2)^3} = 1.$$

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8282. (The Editor.)—If a coin be dropped at random over a grating composed of parallel equidistant wires in a horizontal plane, show by a *general* solution, that, according as the distance between the wire is equal to (1) the radius, (2) the diameter, (3) the circumference of the coin, the respective probabilities that the coin will not go through without striking are $p_1 = \frac{1}{6}\pi + \frac{1}{4}\sqrt{3}$, $p_2 = \frac{1}{4}\pi$, $p_3 = \frac{1}{4}$. 55

8289. (C. Morgan, B.A., R.N.)—If a great circle on the earth be divided in any number of equal parts, and l_1, l_2, l_3 ... be the colatitudes of the points of section, prove that $\Sigma\cos^{-1}\left(\frac{\cos l_n - \cos l_{n-2}\cos 2l_{n-1}}{\sin l_{n-2}\sin 2l_{n-1}}\right) = 2\pi$. 48

8295. (Belle Easton.) — Three bodies A, B, and C, attracting each other according to Newton's law, continue during their motion in a straight line with each other; show that, if C is the middle body,

$$Aa^3(b^3-c^3) + Bb^3(c^3-a^3) = Cc^3(a^3-b^3),$$

where $a = BC$, $b = CA$, $c = AB$, all taken positively. 42

8325. (D. Biddle.)—A is the apex of a tetrahedron ABCD (regular or otherwise) which stands on a horizontal plane, and E is a point on the edge AB. A fly descends from E by making a circuit of the tetrahedron and finally arriving at B. Find at what points it must cross AC and AD, so as to maintain the same gradient to the horizon the whole way down. 58

8338. (Satis Chandra Rây, B.A.)—Solve the equations

$$\sin(\theta+\phi) = a+b, \quad a\tan\phi = b\tan\theta. \text{ ..... } 110$$

8339. (Rev. T. R. Terry, M.A.)—If $\phi(p, q)$ be a function of p and q defined by the equations

$$(2q-2p-1)\phi(p, q) = \phi(p, q-1) = -(p+1)\phi(p+1, q),$$

and if $\phi(1, 0) = 1$, find the value of $\phi(p, q)$. 35

8356. (Professor G. B. Mathews, B.A.)—Prove that (1)

$$\frac{k+(1-ki)e^{\theta i}}{1+ki+ke^{\theta i}} = e^{\phi i}, \text{ where } \tan\phi = \frac{\sin\theta - 2k\cos\theta - 2k^2(1+\sin\theta)}{\cos\theta + 2k(1+\sin\theta)};$$

and (2), $$\tan\tfrac{1}{2}(\theta-\phi) = \frac{k(1+\sin\theta)}{1+k\cos\theta}.$$ 47

8358. (The Editor.) — If the respective distances of the in-centre from the vertices of a triangle be x, y, z, prove that

$$a^4x^4 + b^4y^4 + c^4z^4 + (a+b+c)^2x^2y^2z^2 = 2(b^2c^2y^2z^2 + c^2a^2z^2x^2 + a^2b^2x^2y^2) \text{ .... } 106$$

8361. (Capt. H. Brocard. Suite au Quest. 8326.)—Trouver l'expression des distances des points d'intersection précités aux symédianes. ... 43

8365. (W. J. C. Sharp, M.A.)—Prove that (1) every unicursal twisted quartic curve lies on a quadric surface; (2) every unicursal twisted sextic on a cubic surface; and (3) every unicursal twisted octavic on two quartic surfaces. 42

8369. (Rev. J. J. Milne, M.A. Connected with Professor Crofton's Question 8310.)—An event happens on an average once a year. Show that the chance that it will not happen in any particular year is $1/\epsilon$. 43

8376. (F. Morley, B.A.) — There are two points whose distances

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from fixed points A, B, C are in the ratios $p : q : r$; show that the line joining them goes through a fixed point, whatever be the values of the ratios. 47

8379. (B. Hanumanta Rau, B.A.)—A′B′C′ is a triangle inscribed in another triangle ABC and similar to it. Prove that (1) the orthocentre of A′B′C′ is fixed and coincides with the circum-centre of ABC; (2) the locus of its circum-centre is a fixed straight line through the nine-point centre of ABC; (3) the locus of the centroid is a straight line through the centroid of ABC; (4) the envelopes of the sides A′B′, B′C′, C′A′ are three parabolas touching the sides of ABC two and two; (5) the circum-circles of the triangles AC′B′, BC′A′, CA′B′, and A′B′C′ are all equal; (6) if O_1, O_2, O_3 be the centres of the first three circles, the triangles O_1, O_2, O_3 and A′B′C′ are equal in all respects and have the same nine-point circle... 91

8381. (John Griffiths, M.A.)—If n be a number > 1 and s a number from 1 to $n-1$, show that $1 + 2\Sigma(-)^s \cos^2(s\pi/2n) = 0$. 80

[The relation in Question follows from the elliptic function formula $y = \sin(L + A + B + C + \ldots)$. See *Proceedings of the London Mathematical Society*, Vol. XVI., No. 238.]

8386. (R. Knowles, B.A.)—Tangents TP, TQ meet an ellipse in P and Q; tangents at R, R′, the ends of a diameter, meet TP, TQ in M, N, M′, N′ respectively; if O, O′ be the mid-points of MN, M′N′, prove that OO′, PQ, RR′ are concurrent. 103

8392. (Professor Catalan.)—Démontrer que tout carré impair (supérieur à 9) argumenté de 5 est la somme de quatre carrés consécutifs; et réciproquement. 61

8410. (Rev. T. R. Terry, M.A.)—If $\phi(x) = x^x$, show that

$$\{\phi(1)\,\phi(3) \ldots\ldots \phi(2n-1)\}^{1/n} > \phi(n). \ldots\ldots 28$$

8415. (B. Hanumanta Rau, B.A.)—If A′, B′, C′ and A, B, C are the angles of a triangle, prove that

$$\left(\frac{\sin B'}{\sin B}\right)^2 + \left(\frac{\sin C'}{\sin C}\right)^2 - 2\frac{\sin B' . \sin C'}{\sin B \,.\, \sin C}\cos(A + A')$$
$$: \left(\frac{\sin C'}{\sin C}\right)^2 + \left(\frac{\sin A'}{\sin A}\right)^2 - 2\frac{\sin C' . \sin A'}{\sin C \,.\, \sin A}\cos(B + B') = \sin^2 A : \sin^2 B.$$

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8421. (P. C. Ward, M.A.)—Show that the result of eliminating x from the equations $a(x+1)^n = (-1)^n c$, $ax^n = b$, is $a^{1/n} + b^{1/n} + c^{1/n} = 0$. 116

8426. (Professor Mahendra Nath Ray, M.A., LL.B.)—Show that the series

$$\tan x + \tfrac{1}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x + \ldots \text{ to } \infty = \tfrac{1}{4}\sqrt{(-1)}\sec^{-1}(\cos 2x) \ldots. 35$$

8428. (L. J. Rogers, B.A.)—Prove that, if $\alpha, \beta \ldots$ be any positive quantities, and $s_r \equiv \alpha^r + \beta^r + \ldots$, then $s_m s_r > s_n s_p$, provided

$$m > n > p > r \quad \text{and} \quad m + r = n + p. \ldots\ldots 121$$

8438. (Professor Mathews, M.A.)—If P and Q are any two points on the circle circumscribing a triangle ABC, prove that the angles of intersection of the pedal lines of P and Q, with regard to the triangle ABC, are equal to the angles which PQ subtends at the circumference of the circle. 105

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8439. (Professor Clarke, F.R.S.)—A fine string attached to a fixed point A carries a small ring of mass m, and, passing over a small pulley B in the same horizontal plane with the fixed point, has a mass (m_1+m_2) attached to the free extremity. The system being in equilibrium, the mass m_2 is removed; show that the strain on the fixed point is instantly reduced by $\frac{m_2(m_1+m_2)}{(m_1+m_2)^2+mm_1}$ of its former value. 6

8440. (Professor Nilkanta Sarkar, M.A.) — A, B, C are the three points on a horizontal plane at which the summits of three hills appear to be two and two in the same direction. If x, y, z be the heights of the hills, prove that $\frac{x(y-z)}{BC} = \frac{y(z-x)}{CA} = \frac{z(x-y)}{AB}$. 95

8444. (The Editor.)—If four points P, Q, R, S be taken at random, one in each side of a quadrilateral ABCD, find (1) the probability that the area of PQRS will be less than n (area of ABCD), where n is a proper fraction; (2) apply the result to the case of a rectangle, and therefrom (3) show that the probability that PQRS will be less than three-fourths (or, what is the same thing, greater than one-fourth) of ABCD is

$$\tfrac{1}{2}+\tfrac{5}{4}\log_e 2 = \cdot 991434. \ldots\ldots 87$$

8449. (R. Tucker, M.A.)—Draw the Lemoine-lines DKE′, D′KF, EKF′, *i.e.*, parallels to the sides, through the Lemoine-point (K) of a triangle, and show that (1) K is the centroid of the Lemoine-points of the triangles BD′F, CDE′, AF′E; (2) if Ω, Ω' are the Brocard-points, and ρ_1, ρ_1', &c. the circum-radii of BΩC, BΩ'C, &c., then $\rho_1\rho_2\rho_3 = R^3 = \rho_1'\rho_2'\rho_3'$; (3) if A′B′C′ is the first Brocard-triangle (see Casey's *Conics*), AA′, BB′, CC′ intersect on Kiepert's hyperbola (*cf.* Casey), where it is met by Pπ (see Quest. 7819, P being the orthocentre); (4) the triangle formed with sides equal to the median lines of ABC has the same Brocard-angle as ABC, and, if A_1, B_1, C_1 are the angles of this triangle,

$$\cot A + \cot A_1 = \cot B + \cot B_1 = \cot C + \cot C_1 = \tfrac{3}{2}\cot\omega,$$

and its Brocard-radius is to the Brocard-radius of ABC as $m_1m_2m_3$ is to $3abc$, where m_1, m_2, m_3 are the aforesaid median lines; (5) if A″B″C″ be the second Brocard-triangle, Δ'' its area, and ρ_1', R the Brocard and circum-radius respectively of the triangle formed by the median lines as above, $\Delta''R_1^4/\Delta^2\rho_1'^2 \propto \cot^3\omega$. 40

8450. (Rev. T. R. Terry, M.A.) — The sum of the squares of the coefficients in the expansion of $(1+x+x^2)^n$, where n is a positive integer, is $\Sigma\frac{2n!}{r!\,r!\,(2n-2r)!}$, where r may have any value from 0 to n inclusive. 123

8451. (Rev. J. J. Milne, M.A. General case of Quest. 8266.) — If a centre of similitude of two circles be joined with the point of contact of one of the circles with either common tangent through the other centre of similitude, the line joining the middle point of the line so drawn and the centre of the circle bisects that common tangent. 63

8454. (B. Hanumanta Rau, B.A.)—A balloon rises vertically with uniform acceleration; two minutes after starting, when it has attained a height of 1,600 yards, a stone is projected from it horizontally, with a velocity of 32 yards per second; show that the stone will strike the

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earth at a point 640 yards from the point from which the ascent was made. 80

8460. (J. O'Byrne Croke, M.A. Suggested by Quest. 8430.)—Those sectors of an ellipse are equal in area which are included between radii m and n, and a diameter o, drawn respectively parallel to two tangents from a point P, and their chord of contact. 122

8462. (L. J. Rogers, B.A.)—If (1) $u+v+w = K$, show that

$$Zu+Zv+Zw = \frac{k^2}{k'} \operatorname{cn} u \operatorname{cn} v \operatorname{cn} w ;$$

and if (2) $u+v+w = K+iK'$, then

$$Zu+Zv+Zw+\frac{i\pi}{2K} = \frac{i}{k'} \operatorname{dn} u \operatorname{dn} v \operatorname{dn} w. \text{ ..... } 39$$

8464. (J. Brill, M.A.)—TP and TQ are tangents to a parabola whose focus is S; H and K are the centres of circles passing through S and touching the parabola at P and Q respectively; and L and M are the centres of the circumscribing circles of the triangles SPT and SQT. O is the mean centre of H, K, L, M for multiples m, m, $3m$, $3m$ respectively. Prove that the circle described with centre O and radius OS touches the parabola at the extremity of the diameter through T. 108

8465. (D. Edwardes.)—If $S_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+1/n$, prove that

$$S_1-\left(\frac{1}{2}\right)^2\frac{S_2}{2}+\left(\frac{1.3}{2.4}\right)^2\frac{S_3}{3}-\left(\frac{1.3.5}{2.4.6}\right)^2\frac{S_4}{4}+\ldots=\frac{16}{\pi}(1-\log_e 2)\ldots 75$$

8468. (Professor Bordage.)—A point M is given in the plane of a given triangle ABC. The lines MA, MB, MC are drawn. Prove that the nine-point circles of the triangles MAB, MBC, MCA intersect in a point R lying on the nine-point circle of the triangle ABC. If the point M is on the circum-circle of the triangle ABC, prove that (1) the Simson-lines relative to the triangles MAB, MBC, MCA, ABC and to the respective points C, A, B, M, pass through the point R; (2) the centroids of the four triangles MAB, MBC, MCA, ABC are upon the same circumference. 33

8482. (W. J. M'Clelland, M.A.)—If ABC be a spherical triangle, O the orthocentre, prove that (1) the radius ρ of the polar circle is given by the equation $-\tan^2\rho = (\tan a \tan b \tan c \cos A \cos B \cos C) / 2N$; and deduce the analogous expression *in plano* (see TOWNSEND's *Modern Geometry*, Art. 168; $2N = \sin B \sin C \sin a = \sin C \sin A \sin b = \sin A \sin B \sin c$); (2) the polar circles of the triangles BOC, COA, and AOB have their poles respectively at A, B, C and intersect orthogonally; and (3) the perpendiculars of the triangle ABC are the radical axes of the circles taken two and two, and the orthocentre O is the radical centre of the three polar circles of BOC, COA, and AOB. 117

8484. (C. E. McVicker, M.A.)—If there be given the centre O of a conic that touches the sides of a triangle the mid-points of whose sides are A', B', C', prove that, if OA', OB', OC' intersect B'C', C'A', A'B' in X, Y, Z, then AX, BY, CZ will meet the sides in the points of contact. 96

8487. (J. O'Byrne Croke, M.A.)—If two plane sections of a right cone, S_ϕ, S_ψ, pass through the same point of the axis, and be inclined to

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it respectively at angles θ and ϕ, prove that the radius vector ρ_θ drawn from this point as centre in S_ϕ, and making an angle θ with the transverse axis of the section, is equal to ρ_ϕ in S_θ. 105

8491. (Rev. T. R. Terry, M.A.)—If a series $S \equiv a_0 + a_1 x + a_2 x^2 + \ldots$ is divergent when $x = 1$, find the conditions that $(1-x)\,S$ may vanish when $x = 1$. 95

8499. (Emile Vigarié.)—Dans un triangle ABC on mène des parallèles à la base BC. Démontrer que les diagonales des trapèzes ainsi formés se coupent sur la médiane AM du triangle donné. 34

8500. (D. Edwardes.)—Prove that

$$\int_0^1 \int_0^{\frac{1}{2}\pi} \frac{\log(1-k)}{(1-k\sin^2\phi)^{\frac{1}{2}}}\, dk\, d\phi = 8\log 2 - 8. \quad \ldots\ldots 75$$

8510. (Ch. Hermite, LL.D., Membre de l'Institut.)—Trouver l'expression la plus générale d'un polynôme $F(x)$, de degré $2m$, tel qu'on ait :

$$x^m F\left(\frac{1}{x}\right) = F(x), \quad (x+1)^{2m} F\left(\frac{1-x}{1+x}\right) = 2^m F(x) \quad \ldots\ldots (1, 2).$$

..... 50

8512. (Professor Bordage.)—Construct a quadrilateral, having given the mid-points of two opposites sides, the angle of intersection of the diagonals, the sum of the diagonals, the ratio of two opposite sides, and the sum of the squares of the other two sides. 59

8513. (Professor Neuberg.) — Si $x = a$, $y = b$, $z = c$ est une solution de l'équation $(x+y+z)^3 = 4(x^3+y^3+z^3)$, une seconde solution est $x = a(-a+b+c)$, $y = b(a-b+c)$, $z = c(a+b-c)$. 25

8514. (Professor Mahendra Nath Ray, M.A., LL.B.)—Find the values of x, y, z, u, v, w from the following equations :—

$$yz - u^2 = a^2, \quad zx - v^2 = b^2, \quad xy - w^2 = c^2, \quad vw - xu = d^2,$$
$$wu - yv = e^2, \quad uv - zw = f^2. \quad \ldots\ldots 78$$

8516. (Professor Sircom, M.A.) — In the cubic of Quest. 8396, prove that (1) the tangent at the centre (the real point of inflexion) passes through the Symmedian-point of the triangle ; (2) the tangents at the vertices meet in a point ; (3) if we take the point on the curve at the other extremity of the diameter through the orthocentre, and from it draw lines to the vertices, these will meet the sides of the triangle in points which are the remaining points of section by the curve ; (4) the tangents at the excentres meet in pairs on the lines of (3), also each of these tangents meets the tangent at the incentre on one of these lines..... 115

8519. (Professor Genese, M.A.) — If a and b be positive quantities greater than 1, and $a > b$, show that (1) $a^b > b^a$ if a and b be each $< e$ the Napierian base ; (2) $a^b < b^a$ if a and b be each $> e$; and (3) if e lie between a and b either inequality may occur. 27

8522. (The Editor.) — If one corner of a sheet of paper be folded down so as to lie on (1) a given straight line, (2) a given circle, show that the crease will touch in (1) a parabola whose focus is the given (unfolded) corner, and directrix the given line, and in (2) an ellipse or hyperbola, according as the corner is within or without the given circle. 21

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8524. (R. A. Roberts, M.A.)—Show that the boundary of the solar shadow on the plane z of Sir Robert Ball's cylindroid

$$z(x^2+y^2)-2mxy=0$$

is a tricuspidal hypocycloid; and find the coordinates of its centre in terms of the position of the sun. 32

8526. (Rev. J. J. Milne, M.A. Connected with Quest. 8400).—In the base BC of a triangle ABC determine geometrically the position of the point P such that, if PQ, PR, containing a given angle, meet AC, AB in Q, R, the area PQAR may be the greatest maximum. 107

8528. (Rev. T. C. Simmons, M.A.)—If K be the Lemoine-point of a triangle whose Brocard-points are ω, ω', and whose circumcircle and Brocard-ellipse are denoted respectively by Σ and Σ'; prove that (1) if *any* line be drawn through K, its poles with respect to Σ and Σ' coincide; and hence (2) that Σ and Σ' can be projected into two conics having the projection of K for a common focus and the projection of the polar of K for a common directrix, and whose eccentricities are respectively equal to $2\rho'/R$ and ρ'/R. Prove also (3) that, if Σ' revolve round K through an angle of 180°, its common chord with Σ orthogonally trisects the distance between K and the polar of K; (4) that the lines $K\omega$, $K\omega'$ form the diagonals of a quadrilateral inscribed in Σ, and touching Σ' at the extremities of the minor axis; (5) that the chord of Σ through K parallel to $\omega\omega'$ is quadrisected by Σ'. 30

8529. (J. J. Walker, F.R.S.)—A homogeneous pyramid ABCDE stands upon a quadrilateral base, the diagonals BD, CE of which meet in O; show that the distance of its mass-centre from the face ABC is equal to 3 (Vol. ABCDE − Vol. ADEO) / 4 area ABC. 60

8532. (Emile Vigarié.)—Dans un triangle ABC on donne un côté BC en grandeur et position. Démontrer que, lorsque la somme des deux autres côtés est donnée, (1) La projection de la bissectrice intérieure de l'angle A sur l'un des côtés de cet angle est constante; (2) Le produit des perpendiculaires abaissées des sommets B et C sur la bissectrice extérieure du même angle est constant. 84

8533. (Maurice D'Ocagne.)—Si du point P on voit les coniques H et K sous les angles droits, c'est-à-dire, si le point P est à la rencontre des cercles orthoptiques des coniques H et K, et si la polaire du point P par rapport à la conique K coupe en A et en B la polaire réciproque de la conique H par rapport à la conique K, l'angle APB est droit. 86

8538. (Rev. A. F. Torry, M.A.)—Prove that (1) the tangent to $27ay^2=4(x-2a)^3$ where it meets $y^2=4ax$ is normal to both curves; and (2) the ratio in which this straight line is divided by the curves is

$$8a-\tfrac{1}{2}(7a) : \tfrac{1}{2}(7a)-2a = 3 : 1. \text{ ..... } 123$$

8539. (P. C. Ward, M.A.)—If $p\cos\frac{1}{2}(\alpha+\beta)+q\sin\frac{1}{2}(\alpha+\beta)=k\cos\frac{1}{2}(\alpha-\beta)$; show that

$$\frac{d\alpha}{[p^2+q^2+k^2-2pk\cos\alpha-2qk\sin\alpha]^{\frac{1}{2}}}=\frac{d\beta}{[p^2+q^2+k^2-2pk\cos\beta-2qk\sin\beta]^{\frac{1}{2}}}.$$

..... 70

8541. (C. E. McVicker, M.A.)—Prove that the normal at any point P on a parabola cuts off from the curve an area equal to eight times the area cut off by the focal chord drawn through P. 71

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8545. (D. Edwardes.)— A uniform rod moves in a vertical plane under the action of gravity, with its lower extremity upon a smooth horizontal plane, and its upper extremity constrained to move upon a smooth curve. If the rod starts from rest, prove that it reaches the horizontal position in the shortest time when the curve is an arc of an ellipse of eccentricity $\frac{1}{2}(\sqrt{3})$ and latus rectum equal to half the length of the rod; and, assuming that the rod in its initial position is not to cut the curve, show (2) that there are two ellipses satisfying the condition, if the tangent of the inclination of the rod to the horizon in its initial position is less than $\sqrt{2}$, the time being the same for both curves..... 123

8547. (A. Gordon.)—Prove that (1) the locus of points for which the sum of the squares of the normals to the parabola $y^2 = lx$ is constant $(=c^2)$ is the ellipse $3y^2 + (x+l)^2 = \frac{1}{2}l^2 + c^2$; and (2) the corresponding locus for the general equation of the second degree $(x, y, 1)^2 (a, b, c, f, g, h$ is a conic. 124

8551. (Professor Bordage.) — Construct an inscriptible quadrilateral ABCD, knowing (1) the diagonal BC; (2) the sum AB + AC of the sides AB and AC; (3) the difference of the angles ABC, ACB; (4) the difference of the perpendiculars drawn from the points C and B on the diagonal AD. 23

8552. (Professor Neuberg.)—On donne un angle XOY et une courbe (C). Une tangente quelconque menée à (C) détermine avec l'angle XOY un triangle OAB. Trouver géométriquement la tangente menée au lieu décrit par l'orthocentre ou par le centre de gravité du triangle OAB.... 98

8553. (Professor Mahendra Nath Ray, M.A., LL.B.) — If x, y, z denote the distances of the orthocentre from the angular points of the triangle ABC, prove that

$$xyz(ax+by+cz)^3 + abc(x^2+y^2+z^2)(ax+by+cz)^2 - 4a^3b^3c^3 = 0. \ldots 60$$

8555. (Professor Genese, M.A.)—Any two circles are polar reciprocal with respect to four conics having their centres at the centres of similitude of the circles. The four conics are imaginary if the circles lie outside each other; two of the conics are real when one of the circles lies wholly within the other, and in that case their centre is the internal centre of similitude. Prove also that, if a and b be the usual semi-axes of one of the real conics, a is at right angles to the line of centres of the circles, and a^2/b is a mean proportional between the radii of the circles. 68

8556. (J. J. Walker, F.R.S.) — If S denote the result of substituting the coordinates of a point P in the general equation of a central conic, prove that S is proportional to the square of the area of the quadrilateral formed by the tangents from P to the conic, and the central radii to the point of contact: viz., in Cartesian coordinates, $(\text{area})^2 = -\Delta S / C^2$ 69

8560. (Ch. Hermite, LL.D., Membre de l'Institut.)—Prouver que l'intégrale

$$A = \int_0^\infty \frac{dx}{x}\left\{\frac{1}{(1+x)^n} - \frac{1}{(1+x/a)^n}\right\}$$

a toujours la même valeur quel que soit l'exposant n, à savoir $\log a$. ... 21

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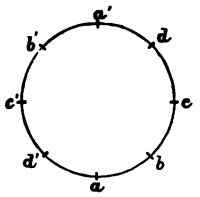
8561. (Professor Catalan.)—Soient huit quantités disposées circulairement. On ajoute deux termes consécutifs, et l'on élève au carré ; on retranche deux termes dont les rangs diffèrent de trois unités, et l'on élève au carré ; on fait la somme de trois termes consécutifs ; on en retranche le terme suivant, et l'on élève au carré. Cela posé : la somme des seize premiers carrés, ainsi formés, égale la somme des huit derniers. De plus, cette dernière somme peut être obtenue de deux manières, à cause des deux sens de rotation. Exemple :



$$\begin{array}{ccc} & 1 & \\ 6 & & 3 \\ 7 & & 2 \\ 4 & & 5 \\ & 8 & \end{array}$$

$$4^2+5^2+7^2+13^2+12^2+11^2+13^2+7^2 = 742;$$
$$4^2+5^2+2^2+2^2+2^2+3^2+4^2+4^2 = 94;$$
$$1^2+2^2+11^2+10^2+13^2+16^2+11^2+8^2$$
$$= 10^2+9^2+14^2+15^2+12^2+9^2+0^2+3^2 = 836.$$
$$742+94 = 836. \quad \text{..... 24}$$

8568. (Rev. T. R. Terry, M.A.) — Three equal circles (radius a) pass through a point O and intersect two and two in the points A, B, C. Three circles are drawn each passing through O and one of the points A, B, C, and cutting the third circle orthogonally in the points D, E, F. Show that (1) the circles AD, BE, CF have a common radical axis ; and (2) the circle drawn through the points D, E, F touches the circle whose centre is O and radius $2a$, and which touches the first three circles. ... 75

8570. (W. W. Taylor, M.A.)—Prove that the coefficient of x^{2n} in the expansion of $(1+x+x^2)^{2n}$ is $\left(\frac{2n!}{n!\,n!}\right)^2 + 2\Sigma\left(\frac{2n!}{r!\,(2n-r)!}\right)^2 \cos\frac{2(n-r)\pi}{3}$, where r can have any value from 0 to $n-1$. 101

8572. (S. Roberts, M.A.)—Show that the centre of the circle bisecting three given circles, whose radii are r_1, r_2, r_3, is the radical centre of three imaginary circles, concentric with the given ones, but whose radii are ir_1, ir_2, ir_3 respectively, and is situated on the right line connecting the centre of the circle through the three given centres and the radical centre of the three given circles, the linear segment between the radical centres being bisected by the centre of the circle through the three given centres. An analogous construction holds for four given spheres. ... 25

8576. (J. J. Walker, F.R.S.)—Show that the axis of the right cone through the (unit) vectors α, β, γ coincides with the vector

$$\Sigma\alpha\,(1+S\beta\gamma)(1-S\beta\gamma+S\gamma\alpha+S\alpha\beta). \quad \text{..... 32}$$

8581. (Maurice D'Ocagne.) — Par un point M, pris d'une manière quelconque sur le côté BC du triangle ABC, on mène des parallèles aux côtés AC et AB. Ces droites coupent respectivement aux points B′ et C′ les côtés AB et AC. Démontrer que, si la droite qui joint le sommet A au point de rencontre I des droites B′C et BC′ coupe la droite B′C′ au point M′, on a $\frac{M'B'}{M'C'} = \frac{MB}{MC}$. 101

8583. (J. Brill, M.A.)—If p denote the pressure at any point in a homogeneous incompressible fluid which is in irrotational motion of a two-dimensional character, subject to the action of a field of force whose

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potential satisfies LAPLACE's equation, prove that

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\left\{\log\left(\frac{d^2p}{dx^2}+\frac{d^2p}{dy^2}\right)\right\}=0. \quad \ldots\ldots\ldots\ 74$$

8584. (S. Tebay, B.A.)—To draw a straight line which shall be trisected by two given circles. 65

8587. (Professor Haughton, F.R.S.) — Two equal masses of ice at 32° F. come into collision with equal and opposite velocities, and half their energy is converted into heat, the other half remaining as motion; find what their velocity must be so that the first half shall become steam at 212° F. 59

8588. (Ch. Hermite, LL.D., Membre de l'Institut.)—Déterminer la valeur de l'intégrale $J=\int_0^{\frac{1}{2}\pi}\cot(x+ai)\,dx$, a désignant une quantité réelle.... 63

8590. (Professor Bordage.)—An isosceles triangle ABC being given, find in its base BC two points D and E, and in the sides AB, AC two points F, G, such that the pentagon AFDEG shall have equal sides. ... 77

8593. (Professor Wolstenholme, M.A., Sc.D.) — The chord PQ of the surface $ax^2+by^2+cz^2=1$ (where $a+b+c=0$) is normal at P; prove that PQ is the harmonic mean between the two principal radii of curvature at P. 65

8594. (Professor Clarke, M.A.) —A particle projected from the lowest point of a vertical circular hoop leaves the arc at a point P, and after moving freely in a parabola meets the circle again in P′, the other end of the diameter through P: prove that PP′ must make an angle $\frac{1}{4}\pi$ with the horizon, and that the space due to the velocity of projection is $(1+\frac{3}{4}\sqrt{2})\times$ radius of the circle. 102

8595. (Professor Hudson, M.A.)—Prove that

$$\int_0^{\pi}(A_0+A_1\cos\theta+A_2\cos^2\theta+\ldots A_n\cos^n\theta+\ldots)\,d\theta$$
$$=\pi\Sigma A_{2n}\frac{(2n-1)(2n-3)\ldots 1}{2n(2n-2)\ldots 2}. \quad \ldots\ldots\ldots\ 78$$

8596. (Professor Hadamard. Généralisation de la Quest. 8510.)—Trouver (1) les polynômes $F(x)$, de degré $2p$, tels que

$$(c+bx)^{2p}F\left(\frac{a-cx}{c+bx}\right)\equiv(c^2+ab)^pF(x),$$

et

$$(c'+b'x)^{2p}F\left(\frac{a'-c'x}{c'+b'x}\right)\equiv(c'^2+a'b')^pF(x).$$

En général, ces polynômes sont de la forme $(Ax^2+Bx+C)_p$, A, B, C étant des constantes déterminées. Examiner (2) le cas (réalisé dans la question 8510) où il en est autrement, et voir comment doivent être choisies à cet effet les constantes a, b, c, a', b', c'. 73

8597. (Professor Ignacio Beyens.)—Dans un triangle sphérique qui a un angle $A=60°$, on vérifie la relation entre les autres éléments du triangle, étant égal $a+b+c$,

$$\sin B\sin C=3\sin(s-b)\sin(s-c)\,/\sin^2 a,\ 2s. \quad \ldots\ldots\ 82$$

8601. (Professor Byomakesa Chakravarti, M.A.)—A lamina bounded by a cycloid and its base has its centre of gravity at the middle point

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of its axis. It is placed with its base vertical on a perfectly rough horizontal plane, and allowed to roll down: show that, at the moment its vertex reaches the ground, its angular velocity is $\left\{\frac{2ag(\pi-1)}{a^2+k^2}\right\}^{\frac{1}{2}}$, where a is the radius of the generating circle, and k the radius of gyration about the centre of gravity. 72

8602. (The Editor.)—If three points are taken at random, one on each side of a given triangle, find the average area of the circle drawn through these three points. 118

8605. (Rev. T. P. Kirkman, M.A., F.R.S.)—If, in any 4-edron $abcd$, ea be the 2-edral angle about ac, prove that

$$\cos\widehat{ea}+\cos(\widehat{cb}-\widehat{cd})\,[(bd)^2-(eb-ed)^2]$$
$$=[\cos\widehat{ea}+\cos(\widehat{ad}-\widehat{ab})]\,[(bd)^2-(ad-ab)^2]. \ldots 62$$

8606. (Asûtosh Mukhopâdhyây, M.A., F.R.A.S.)—Two sources of heat are placed at the foci of an ellipse, the strength of one being m times that of the other. Show that the points on the periphery at which a person would feel the maximum quantity of heat are determined as the intersection of the ellipse with the line $x=\frac{a}{e}\cdot\frac{1-m^{\frac{1}{2}}}{1+m^{\frac{1}{2}}}$. 72

8608. (S. Roberts, M.A.) — Show that, if a given triangle ABC be taken as the triangle of reference, and straight lines be drawn through any point on the line $\alpha\cos A+\beta\cos B+\gamma\cos C=0$, parallel to the sides of the given triangle, the six finite intersections with the sides lie on a rectangular hyperbola. 70

8609. (R. Tucker, M.A.)—Prove that

$$\begin{vmatrix} -b^2c^2, & ab(a^2+c^2), & ac(a^2+b^2) \\ ab(b^2+c^2), & -c^2a^2, & bc(a^2+b^2) \\ ac(b^2+c^2), & bc(a^2+c^2), & -a^2b^2 \end{vmatrix} = (a^2b^2+b^2c^2+c^2a^2)^3. \ldots 69$$

8613. (Captain H. Brocard.)—D et D′ étant les points dont les distances sont inversement ou directement proportionnelles aux cubes des côtés du triangle, on sait que D est le centre d'homologie du triangle donné ABC et du premier triangle de Brocard A′B′C′, et que D′ est le pôle de ΩΩ′ par rapport au cercle de Brocard. Cela posé, on demande d'établir que le pôle de DD′ relativement au même cercle est sur l'axe d'homologie des triangles ABC, A′B′C′. 99

8614. (Maurice D'Ocagne.)—Les cercles C et C′, dont les centres sont O et O′, sont tangents en A ; soient M un point du cercle C, M′ un point du cercle C′, tels que l'angle MAM′ soit droit. La droite MM′ passe par un point fixe qui est le conjugué harmonique du point A par rapport aux points O et O′. 71

8615. (D. Edwardes.)—The space between two coaxal cylinders is filled with homogeneous liquid, and the outer is surrounded by liquid extending to infinity, the whole being bounded by planes perpendicular to the axis. If the inner be suddenly moved perpendicular to its axis, prove that the outer will begin to move with

$$2b^2c^2\rho\,/\left[\rho(a^2b^2-a^2c^2+b^4+b^2c^2)+\sigma(a^2-b^2)(b^2-c^2)\right]$$

times the velocity of the inner, where a, b are the external and internal radii of the outer cylinder, σ its density, c the radius of the inner cylinder, and ρ the density of the liquid. 103

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8619. (T. A. E. Sanderson.)—Any point O (h, k) is taken in the plane of a parabola $(y^2 = 4ax)$; and through this point three normals OP, OQ, OR are drawn to the curve. If (α, β) be the centre of the circumscribed circle of the triangle PQR, prove that $2\alpha = h + 2a$. 103

8630. (Prince Camille de Polignac.) — Prove that the equation $a^n - 2^k = \pm 1$ is impossible in integers unless $n = 1, 2$. [In the latter case $a = 3$.] 109

8633. (Ch. Hermite, LL.D., Membre de l'Institut.)—Démontrer la formule $D_x \operatorname{sn}(x+a) \operatorname{sn}(x+b) = \dfrac{\operatorname{sn}^2(x+a) - \operatorname{sn}^2(x+b)}{\operatorname{sn}(a-b)}$. 94

8634. (Professor Steggall, M.A.)—Show that the general solution in rational quantities of the equation $x^2 + xy + y^2 = z^2$ is

$$x = p^2 - q^2, \quad y = 2pq + q^2, \quad z = p^2 + pq + q^2. \text{ ..... } 97$$

8636. (Professor Mahendra Nath Ray, M.A., LL.B.)— Show that the following equations are satisfied by the *same* value of x, and find this value :—

$$ax(x^2 - a^2)^{\frac{1}{2}} + bx(x^2 - b^2)^{\frac{1}{2}} + cx(x^2 - c^2)^{\frac{1}{2}} = 2abc,$$

$$2(x^2 - a^2)^{\frac{1}{2}}(x^2 - b^2)^{\frac{1}{2}}(x^2 - c^2)^{\frac{1}{2}} = x(a^2 + b^2 + c^2 - 2x^2). \text{ ..... } 49$$

8637. (Professor Hudson, M.A.) — The tangent and normal to $y/c = \frac{1}{2}(e^{x/c} + e^{-x/c})$ cut the axis of x in T, G respectively; find the minimum value of GT. 116

8638. (Professor Ignacio Beyens.) — Diviser (1) un nombre (n) en quatre parties telles que la première ajoutée a un certain nombre, la seconde retranchée le même nombre, la troisième multipliée par ce nombre et la quatrième divisée par ce nombre, donnent le même résultat; trouver (2) une formule pour résoudre ce problème d'une manière générale; et démontrer (3) que le *nombre* de solutions qu'a la question (en nombres entiers et positifs) est égal au *nombre* des diviseurs carrés qu'a le nombre proposé. 112

8640. (Professor Bordage.)—Through a point (P) lying in the plane of a given circle (O), a secant is drawn; (I) is the projection of the centre (O) upon the intercepted chord. Draw OI and take upon that line IM = IP. What is the geometric locus of the point M, when the secant varies? 94

8641. (Professor Neuberg.) — Le cercle osculateur en un point M d'une ellipse rencontre l'ellipse au point M_1; le cercle osculateur en M_1 rencontre l'ellipse en M_2; le cercle osculateur en M_2 la rencontre en M_3, etc.; (1) le polygone $MM_1M_2M_3\ldots$ peut-il se fermer? (2) l'enveloppe de la corde MM_2 est la projection orthogonale d'une épicycloïde. 114

8664. (J. Brill, M.A.)—O is the mean centre of the points A, B, C for multiples l, m, n respectively. Prove that, if OA = ξ, OB = η, OC = ζ, and $2\sigma = l\xi + m\eta + n\zeta$, the area of the triangle ABC is

$$\frac{l+m+n}{lmn}\{\sigma(\sigma - l\xi)(\sigma - m\eta)(\sigma - n\zeta)\}^{\frac{1}{2}}. \text{ ..... } 121$$

8679. (Professor Catalan.) — Soit y une fonction *impaire* de x, telle que la quantité $\{(1+y^2)^{\frac{1}{2}} - 1\}/y$ reste finie, quand (x) varie entre $-a$) et $(+a)$. Cela posé : $\int_{-a}^{+a} \dfrac{dx}{1 + y + (1+y^2)^{\frac{1}{2}}} = a$. 94

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8687. (Rev. T. P. Kirkman, M.A., F.R.S.) — Given any two circles, meeting or not, by which a straight line L through neither centre is cut into three segments in order proportional to A, B, C. Required to draw a second straight line not equal to L, whose three segments so made shall be in the same proportion in the same order. 65

8688. (J. J. Walker, F.R.S.)—If ρ is the radius of curvature at any point of the orbit; r', p' the radius vector from the pole to, and the perpendicular on the tangent at, the corresponding point of the hodograph, show that $f \propto r'^3 / \rho p'$. 105

8689. (S. Roberts, M.A.) — Given two circles C_1, C_2, show that the centres of circles which (1) are orthogonal to C_1 and are bisected by C_2, or which (2) are orthogonal to C_2 and are bisected by C_1, lie on one and the same circle whose centre is at the mid-point between the centres of the given circles, and which is coaxal with them; also (3) the circle orthogonal to three given circles, the circle which bisects them, and the circle through their centres are coaxal. 111

8699. (J. Griffiths, M.A.)—Solve $x^4 + 8x^3 + 6x^2 - 4x - 2 = 0$. 96

8710. (Professor Sylvester, F.R.S.)—For greater brevity, let q, r, s, t denote respectively $\frac{\delta_x^2 y}{2}$, $\frac{\delta_x^3 y}{6}$, $\frac{\delta_x^4 y}{24}$, $\frac{\delta_x^5 y}{120}$. Prove that the general integral of the differential equation

$$(5q^2t - 15qrs + 10r^3)^2 + (2qs - \tfrac{6}{5}r^2)^3 = 0$$

is $ax + by + c = \log(dx + ey + f)$. 93

8717. (Ch. Hermite, LL.D., Membre de l'Institut.)—Prouver que l'expression $e^{i\pi x/2K} \frac{H^2[x + \frac{1}{2}Ki']}{\Theta^2(x)}$ se ramène à celle-ci: $A \sin x + BD_x \operatorname{sn} x$, A et B étant des constantes. Trouver les valeurs de ces constantes. 111

8733. (W. J. C. Sharp, M.A.) — If the $2n$ points in which a conic cuts an n-ic curve be joined two-and-two by n straight lines, prove that the other $n(n-2)$ points in which these lines intersect the n-ic lie upon an $(n-2)$-ic curve. 97

8764. (Rev. D. Thomas, M.A.)—If $1^2, 2^2 \ldots (n+1)^2$ be the roots of $f(x) = 0$ prove that $\frac{(n+1)^{2n}-1}{f'(n+1)^2} + \frac{n^{2n}-1}{f'(n^2)} + \ldots \frac{1^{2n}-1}{f'(1^2)} = 1$. 93

APPENDIX I.

Unsolved Questions by W. J. C. Sharp, M.A. 125

APPENDIX II.

Circular Notes. By R. Tucker, M.A. 137

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MATHEMATICS

FROM

THE EDUCATIONAL TIMES.

WITH ADDITIONAL PAPERS AND SOLUTIONS.

8560. (By Professor HERMITE, LL.D.)—Prouver que l'intégrale

$$A = \int_0^\infty \frac{dx}{x}\left\{\frac{1}{(1+x)^n} - \frac{1}{(1+x/a)^n}\right\}$$

a toujours la même valeur quel que soit l'exposant n, à savoir $\log a$.

Solutions by (1) Professor CATALAN; (2) Rev. T. R. TERRY, M.A.

1. $A = \int_0^\infty \frac{dz}{z}\{(1+az)^{-n} - (1+z)^{-n}\}$, et, par conséquent,

$$\frac{dA}{da} = -n\int_0^\infty (1+az)^{-n-1}\,dz = \frac{1}{a}; \text{ puis } A = L.a + const.$$

Lorsque $a = 1$, $A = 0$; donc $const. = 0$, et $A = L.a$.

2. In the obvious result $\int_0^\infty \frac{n\,dy}{(1+y)^{n+1}} = 1$, put $y = x/a$, therefore

$$\int_0^\infty \frac{n\,dx}{a^2}\cdot\frac{1}{(1+x/a)^{n+1}} = \frac{1}{a}.$$

Integrating with respect to a from 1 to a, we get the required result.

8522. (By the EDITOR.)—If one corner of a sheet of paper be folded down so as to lie on (1) a given straight line, (2) a given circle, show that the crease will touch in (1) a parabola whose focus is the given (unfolded) corner, and directrix the given line, and in (2) an ellipse or hyperbola, according as the corner is within or without the given circle.

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Solution by Prof. GENESE, M.A.; Rev. T. R. TERRY, M.A.; *and others.*

1. Let the corner O fold to P, then the crease is the perpendicular bisector of OP; therefore it is the locus of points equally distant from P and O. If P′ be another position of P, the intersection of two creases will be the centre of the circle OPP′. Diminishing PP′ indefinitely, the point of contact of the crease with its envelope is the centre Q of the circle through O touching the given line at P. Thus OQ = perpendicular from Q on given line, and therefore the locus of Q is, &c.

2. The same method is available here, and also, in fact, when the line whose envelope is required is an easily recognised locus.

[As an exercise in analysis, let us take the given corner O as origin, and $x\cos\alpha + y\sin\alpha - p = 0$ as given line. The coordinates of any point P on this line are $p\cos\alpha - \lambda\sin\alpha$ and $p\sin\alpha + \lambda\cos\alpha$. If P be made to coincide with O, the crease which bisects OP at right angles is

$$\frac{2x - p\cos\alpha + \lambda\sin\alpha}{2y - p\sin\alpha - \lambda\cos\alpha} + \frac{p\sin\alpha + \lambda\cos\alpha}{p\cos\alpha - \lambda\sin\alpha} = 0;$$

or $$\lambda^2 + 2\lambda(x\sin\alpha - y\cos\alpha) + p(2x\cos\alpha + 2y\sin\alpha - p) = 0;$$

therefore the envelope is $(x\sin\alpha - y\cos\alpha)^2 = p(2x\cos\alpha + 2y\sin\alpha - p)$,

or $$x^2 + y^2 = (x\cos\alpha + y\sin\alpha - p)^2].$$

7432. (By Professor LUCAS.)—On place n jetons sur les cases d'un échiquier carré de n^2 cases, de telle sorte qu'il n'y ait qu'un seul jeton dans chacune des colonnes ou des lignes de l'échiquier; démontrer que (1) si l'on désigne par S_n le nombre des solutions symétriques par rapport au centre de l'échiquier, on a $S_{2n+1} = S^{2n} = 2 \cdot 4 \cdot 6 \ldots (2n)$; (2) si l'on désigne par u_n le nombre des solutions symétriques par rapport à l'une des diagonales, on a $u^n = u_{n-1} + (n-1)u_{n-2}$, et aussi

$$u^n = 1 + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4} + \frac{n(n-1)\ldots(n-5)}{2 \cdot 4 \cdot 6} + \&c.;$$

et (3) traiter la même question pour les solutions symétriques par rapport aux deux diagonales.

Solution by W. J. CURRAN SHARP, M.A.

Evidently $S_{2n+1} = S_{2n}$; for, taking any admissible arrangement of $2n$ counters on a board of $(2n)^2$ squares, and splitting the board into four equal parts by lines parallel to its sides, and inserting a row and a column (the only way in which a new row and column can be symmetrically inserted), there is only one available position for the new counter, viz., the centre. To derive the admissible arrangements of $2n$ counters from those of $2n$, 2 rows and 2 columns must be symmetrically inserted, and this may be done in $n+1$ ways; the two additional counters may then be placed at the new squares on either diagonal, therefore $S_{2n+2} = 2(n+1)S_{2n}$, and, since $S_2 = 2$, evidently $S_{2n} = 2 \cdot 4 \ldots (2n)^2$.

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If v_n denote the number of arrangements symmetrical with respect to a diagonal where none of the counters lie on the diagonal, it will be seen, by erasing the rows and columns which contain the occupied squares on the diagonal (v_n being evidently zero for all odd values of n), that u_n includes :—

1 solution with all the counters on the diagonal,

$\frac{n(n-1)}{1.2} v_2$ solutions with all the counters but 2 on the diagonal,

$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}$ solutions with all but 4 on the diagonal,

so that ultimately

$$u_n = 1 + \frac{n(n-1)}{1.2} v_2 + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4} v_4 + \&c.$$

By inspection $v_2 = 1$.

In v_4 the only available positions for the two which must lie on one side of the diagonal are the intersections of the rows and columns which contain the vacant squares on the diagonal. Now there are three such positions on any one column each of which may be combined with the v_2 positions available when all the occupied rows and columns are erased; therefore $v_4 = 3v_2 = 3$, similarly $v_6 = 5v_4 = 3 \cdot 5$, and so on ; therefore

$$u_n = 1 + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)(n-3)}{2.4} + \frac{n(n-1)\ldots(n-5)}{2.4.6} + \&c.,$$

which is easily shown to satisfy the equation of differences

$$u_n = u_{n-1} + (n-1)\, u_{n-2}.$$

Again, if z_n denote the number of solutions symmetrical with respect to both diagonals, evidently, by the same argument as with S, $z_{2n+1} = z_{2n}$, and by trial $z_1 = 1$, $z_2 = 2 = z_3$, $z_4 = 6 = z_5$; and, if w_n denote the number of symmetrical with respect to both diagonals, where no counters lie upon one diagonal, w_n is zero for all values of u, and by inspection $w_2 = 1$, $w_4 = 3$, &c. ; and, since a counter placed in the upper row involves the removal of two rows and columns if placed on the diagonal, and of four if placed elsewhere, therefore $w_{2n} = w_{2n-2} + 2(n-1)\, w_{2n-4}$, the solution of which is

$$w_{2n} = 1 + \frac{n(n-1)}{1} + \frac{n(n-1)(n-2)(n-3)}{1.2} + \frac{n(n-1)\ldots(n-5)}{1.2.3} + \&c.,$$

and

$$z_{2n} = 1 + nw_2 + \frac{n(n-1)}{1.2} w_4 + \&c.,$$

as appears by erasing the occupied rows and columns when $2n$, $2(n-1)$, $2(n-2)$, &c. counters on the one diagonal are disposed symmetrically with respect to the other.

[The equation $u_n = u_{n-1} + (n-1)\, u_{n-2}$ may be at once obtained, for, if a counter be placed in the top row, if it be on the diagonal this involves the erasure of one row and column, but if in any other position that of two, and the equation follows.]

8551. (By Professor Bordage.)—Construct an inscriptible quadrilateral ABCD, knowing (1) the diagonal BC ; (2) the sum AB + AC of the

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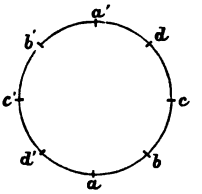
sides AB and AC; (3) the difference of the angles ABC, ACB; (4) the difference of the perpendiculars drawn from the points C and B on the diagonal AD.

Solution by Professor MATHEWS, M.A.

The point A is determined as an intersection of two conics, viz., an ellipse of which B, C are foci, and a rectangular hyperbola of which BC is a diameter. Taking any one of these intersections for A, draw the circle ABC. Now, since the difference of the perpendiculars from B, C on AD is constant, the line AD touches a fixed circle, the centre of which is the middle point of BC. Constructing this circle, and drawing tangents to it from A, the intersections of these tangents with the circle ABC give possible positions of D. Thus, generally speaking, there are eight different solutions.

8561. (By Professor CATALAN.)—Soient huit quantités disposées circulairement. On ajoute deux termes consécutifs, et l'on élève au carré; on retranche deux termes dont les rangs diffèrent de trois unités, et l'on élève au carré; on fait la somme de trois termes consécutifs; on en retranche le terme suivant, et l'on élève au carré. Cela posé: la somme des seize premiers carrés, ainsi formés, égale la somme des huit derniers. De plus, cette dernière somme peut être obtenue de deux manières, à cause des deux sens de rotation. Exemple:



1 6 3 7 2 4 5 8

$$4^2+5^2+7^2+13^2+12^2+11^2+13^2+7^2 = 742;$$

$$4^2+5^2+2^2+2^2+2^2+3^2+4^2+4^2 = 94;$$

$$1^2+2^2+11^2+10^2+13^2+16^2+11^2+8^2$$

$$= 10^2+9^2+14^2+15^2+12^2+9^2+0^2+3^2 = 836.$$

$$742+94 = 836.$$

Solution by the Rev. T. R. TERRY, M.A.

Let a, b, c, d, e, f, g, h be the eight quantities, and s_1, s_2, s_3, s_4 the four sums. Let $\Sigma ab = ab+bc+\ldots$, $\Sigma ac = ac+bd+ce+\ldots$, $\Sigma ad = ad+be+cf+\ldots$ Then, clearly, $s_1 = 2\Sigma a^2 + 2\Sigma ab$, $s_2 = 2\Sigma a^2 - 2\Sigma ad$,

$s_3 = (a+b+c-d)^2+\ldots$, $= 4\Sigma a^2 + 2\Sigma ab + 2\Sigma ac - 2\Sigma ad + 2\Sigma bc - 2\Sigma bd - 2\Sigma cd$.

Now $\Sigma ab = \Sigma bc = \Sigma cd$, and $\Sigma ac = \Sigma bd$, therefore $s_3 = s_1 + s_2$. Also s_1 and s_2 are not changed by reversing the order of the letters, therefore

$$s_4 = s_1 + s_2 = s_3.$$

7998. (By Professor HAUGHTON, F.R.S.)—Four points on a quartic lie on a line (A); three other points lie on a line (B); three other points lie on a line (C); there are (of course) two other real points, lying on (B)

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and (C) respectively; prove that, for every possible quartic passing through the above ten points, the line joining the remaining two real points passes through a fixed point which can be constructed.

Solution by W. J. CURRAN SHARP, M.A.

If A′, B′, C′ be the intersections of B and C, C and A, and A and B, a_1, a_2, a_3, a_4; b_1, b_2, b_3; c_1, c_2, c_3 the ten given points on A, B, and C respectively, and y and z the additional points on B and C; by CARNOT's Theorem we have

$$C'a_1 \cdot C'a_2 \cdot C'a_3 \cdot C'a_4 \cdot A'b_1 \cdot A'b_2 \cdot A'b_3 \cdot A'y \cdot B'c_1 \cdot B'c_2 \cdot B'c_3 \cdot B'z$$
$$= B'a_1 \cdot B'a_2 \cdot B'a_3 \cdot B'a_4 \cdot C'b_1 \cdot C'b_2 \cdot C'b_3 \cdot C'y \cdot A'c_1 \cdot A'c_2 \cdot A'c_3 \cdot A'z,$$

and $\dfrac{A'y \cdot B'z}{C'y \cdot A'z}$ = a constant for all positions of y and z. Now, if A′B′C′ be taken as triangle of reference, and $l\alpha + m\beta + n\gamma = 0$ be the equation to the line yz, $\dfrac{B'z}{A'z} = \dfrac{m \sin A'}{l \sin B'}$ and $\dfrac{A'y}{C'y} = \dfrac{l \sin C'}{n \sin A'}$;

therefore $\dfrac{m \sin C'}{n \sin B'}$ = a constant, or $m = nh$, where h is a constant for all positions of y and z, and the equation to yz is $l\alpha + n(h\beta + \gamma) = 0$,

and therefore the line passes through a fixed point on A, which is easily constructed. [Another Solution is given on p. 101 of Vol. 43.]

8513. (By Professor NEUBERG.) — Si $x = a$, $y = b$, $z = c$ est une solution de l'équation $(x+y+z)^3 = 4(x^3+y^3+z^3)$, une seconde solution est $x = a(-a+b+c)$, $y = b(a-b+c)$, $z = c(a+b-c)$.

Solution by R. F. DAVIS, M.A.; J. YOUNG, M.A.; *and others.*

The relation $(a+b+c)^3 = 4(a^3+b^3+c^3)$ may be written

$$(b+c-a)(c+a-b)(a+b-c)+4abc = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If now $A = a(-a+b+c)$, $B = \ldots$, $C = \ldots$;

then $B+C-A = (c+a-b)(a+b-c)$, &c.,

and $(B+C-A)(C+A-B)(A+B-C)+4ABC$

obviously contains the sinister member of (1) as a factor, and consequently vanishes. Hence $(A+B+C)^3 = 4(A^3+B^3+C^3)$.

8572. (By S. ROBERTS, M.A.)—Show that the centre of the circle bisecting three given circles, whose radii are r_1, r_2, r_3, is the radical centre of three imaginary circles, concentric with the given ones, but whose radii are ir_1, ir_2, ir_3 respectively, and is situated on the right line connecting the centre of the circle through the three given centres and the radical centre of the three given circles, the linear segment between the radical centres being bisected by the centre of the circle through the three given centres. An analogous construction holds for four given spheres.

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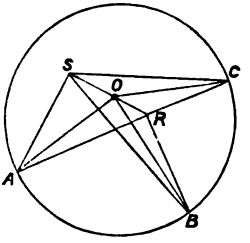
Solutions by (1) Prof. MATHEWS, M.A.; (2) *the* PROPOSER.

1. Let A, B, C be the centres of the given circles; R their radical centre; O the centre of the circle ABC. Produce RO to S, making OS = RO; then $AS^2 + AR^2 = 2(AO^2 + OR^2)$

$= BS^2 + BR^2 = CS^2 + CR^2$.



Now $AR^2 - r_1^2 = BR^2 - r_2^2 = CR^2 - r_3^2$;

therefore $AS^2 + r_1^2 = BS^2 + r_2^2 = CS^2 + r_3^2$.

This shows that S is the centre of the circle bisecting the given circumference; moreover we may write

$AS^2 - (ir_1)^2 = BS^2 - (ir_2)^2 = CS^2 - (ir_3)^2$.

2. *Otherwise*:—If R be the radius of the bisecting circle, and ρ_1, ρ_2, ρ_3 the vectors from its centre to the given centres, we have

$$R^2 - \rho_1^2 - r_1^2 = R^2 - \rho_2^2 - r_2^2 = R^2 - \rho_3^2 - r_3^2 = 0,$$

or the centre is determined by

$$\rho_1^2 - i^2 r_1^2 - (\rho_2^2 - i^2 r_2^2) = \rho_1^2 - i^2 r_1^2 - (\rho_3^2 - i^2 r_3^2) = 0,$$

and if (x, y), (x', y') are coordinates of the two radical centres in the question, we find $\frac{1}{2}(x + x')$, $\frac{1}{2}(y + y')$ are independent of the radii. These are the coordinates, therefore, of the centre of the circle through the three given centres.

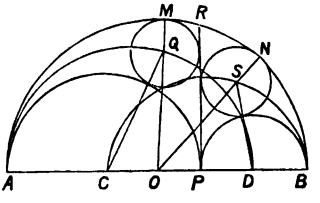
The radical centre of the imaginary circles can be at once found as the intersection of the radical axes of the two pairs of real circles

$$\left.\begin{aligned} \rho_1^2 - r_2^2 &= 0 \\ \rho_2^2 - r_1^2 &= 0 \end{aligned}\right\}, \text{ and } \left.\begin{aligned} \rho_1^2 - r_3^2 &= 0 \\ \rho_3^2 - r_1^2 &= 0 \end{aligned}\right\}.$$

7703. (By W. G. LAX, B.A.)—A, P, B are three points in order in a straight line; on AB, AP, PB as diameters semicircles are described all on the same side of AB. At P a perpendicular PR to AB is erected meeting the semicircle on AB in R; prove that the circles inscribed to the curvilinear figures ARP, BRP have equal radii, and find the value.

Solution by D. BIDDLE; Rev. T. GALLIERS, M.A.; *and others.*

It has already been shown (7449) that the loci of the centres of circles touching others in the way described, are elliptical. In the present case there are two such circles, whose position is further limited by the perpendicular PR. Q, the centre of one, is on the ellipse whose foci are C, O, the mid-points of AP, AB; and S, the centre of the other, is on the ellipse whose foci are D, O, the mid-points of PB, AB. The major axis of the former ellipse is AD, and of the latter BC.



The radii of the circles are thus found:—

$$(OA - MQ)^2 - (OP - MQ)^2 = (AC + MQ)^2 - (AC - MQ)^2,$$

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whence $MQ = \frac{OA^2 - OP^2}{2(OA + 2AC - OP)} = \frac{OA^2 - OP^2}{2AB}$;

and $(OA - NS)^2 - (OP + NS)^2 = (BD + NS)^2 - [OD - (OP + NS)^2]$,

whence $NS = \frac{OA^2 - BD^2 + OD^2 - 2OD \cdot OP}{2(OA + BD + OD)} = \frac{OA^2 - OP^2}{2AB}$;

therefore $NS = MQ = \frac{OA^2 - OP^2}{2AB} = \frac{(OA + OP)(OA - OP)}{2AB} = \frac{AP \cdot PB}{2AB}$.

[This Question is a case of the Arbelon of the ancient Geometers, which has been discussed in some of our earlier volumes.]

8519. (By Professor Genese, M.A.)—If a and b be positive quantities greater than 1, and $a > b$, show that (1) $a^b > b^a$ if a and b be each $< e$ the Napierian base; (2) $a^b < b^a$ if a and b be each $> e$; and (3) if e lie between a and b either inequality may occur.

Solution by the Rev. T. R. Terry, M.A.

If $u = x^{\frac{1}{x}}$, then $\frac{1}{u}\frac{du}{dx} = \frac{1}{x^2}(1 - \log x)$;

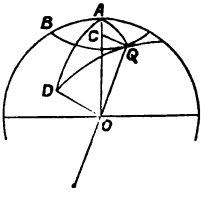
therefore u increases until $x = e$ and then decreases, whence all three results follow obviously.

8137. (By the Editor.)—If the velocity of a carriage along a road is proportional to the cube of the cosine of the inclination of the road to the horizon, determine (1) the path of quickest ascent from the bottom to the top of a hemispherical hill and show that it consists of a spherical curve described by a point of a great circle which rolls on a small circle described about the pole with radius $\frac{1}{6}\pi$, together with an arc of a great circle; and show (2) how the discontinuity is introduced into this problem.

Solution by Professor Sebastian Sircom, M.A.

Let a be the radius of the hill; an element of the curve is given by $ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta\, d\phi^2$. Cosine of angle of inclination

$$= \left\{1 - \left(\frac{dr}{ds}\right)^2\right\}^{\frac{1}{2}} = \left\{1 - a^2 \sin^2\theta \left(\frac{d\theta}{ds}\right)^2\right\}^{\frac{1}{2}}.$$



By the question, we have to make

$$\int \left\{1 - a^2 \sin^2\theta \left(\frac{d\theta}{ds}\right)^2\right\}^{-\frac{1}{2}} ds \text{ a minimum.}$$

By the usual formula $V = Pp + C$, we have, choosing the initial circumstances so that $C = 0$,

$$\left\{1 - a^2 \sin^2\theta \left(\frac{d\theta}{ds}\right)^2\right\}^{-\frac{1}{2}} = 3a^2 \sin^2\theta \left(\frac{d\theta}{ds}\right)^2 \left\{1 - a^2 \sin^2\theta \left(\frac{d\theta}{ds}\right)^2\right\}^{-\frac{1}{2}} \ldots\ldots(1),$$

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a solution of which is

$$\left(\frac{ds}{d\theta}\right)^2 = 4a^2\sin^2\theta \text{ or } s = 2a\cos\theta, \text{ or } \left(\frac{d\theta}{d\phi}\right)^2 = \frac{\sin^2\theta}{3-4\cos^2\theta} \quad \ldots\ldots(2);$$

(2) is satisfied by the curve of the question; for, let ϕ be the azimuth of P from the cusp B, PA $= \theta$, PQ $= \psi$, AQ $= \frac{1}{2}\pi$.

Then arc BQ = arc PQ, therefore $\angle$ BCQ $= 2\angle$ POQ $= 2\psi$. Since AQP $= \frac{1}{2}\pi$, $\cos\theta = \frac{1}{2}\sqrt{3}\cos\psi$, and $\tan\psi = \frac{1}{2}\tan(2\psi-\phi)$; whence, differentiating, we obtain

$$\left(\frac{d\theta}{d\psi}\right)^2 = \frac{3-4\cos^2\theta}{4\sin^2\theta}, \quad \frac{d\phi}{d\psi} = \frac{3-4\cos^2\theta}{2\sin^2\theta}, \text{ which give (2).}$$

When $\theta = \frac{1}{2}\pi$, $\left(\frac{d\theta}{d\phi}\right)^2 = \frac{1}{3}$, giving the initial direction of motion.

When $\theta = \frac{1}{3}\pi$, $\frac{d\theta}{d\phi} = \infty$, and is imaginary for smaller values of θ.

Hence the shortest path, as far as the small circle with radius $\frac{1}{3}\pi$, consists of this curve meeting the circle at a cusp at right angles; the rest of the path is then evidently the arc of the meridian from the small circle to the pole. The discontinuity may be considered as arising from the fact that the analysis shows that the question really includes two distinct problems; the first being to find the path of quickest ascent from the bottom to the small circle, the second from the small circle to the top.

8410. (By Rev. T. R. Terry, M.A.)—If $\phi(x) = x^r$, show that

$$\{\phi(1)\,\phi(3)\ldots\ldots\phi(2n-1)\}^{1/n} > \phi(n).$$

Solution by W. J. C. Sharp, M.A.; W. W. Taylor, M.A.; *and others.*

$$\left(1-\frac{r}{n}\right)\left(-\frac{r}{n}-\frac{r^2}{2n^2}-\frac{r^3}{3n^3}-\&c.\right)+\left(1+\frac{r}{n}\right)\left(\frac{r}{n}-\frac{r^2}{2n^2}-\frac{r^3}{3n^3}-\&c.\right)$$

$$= 2\left\{\frac{r^2}{1\,.\,2n^2}+\frac{r^4}{3\,.\,4n^4}+\frac{r^6}{6n^6}+\&c.\right\}, \text{ and is therefore } > 0,$$

or $$\left(1-\frac{r}{n}\right)\log\left(1-\frac{r}{n}\right)+\left(1+\frac{r}{n}\right)\log\left(1+\frac{r}{n}\right) > 0,$$

therefore $$(n-r)\log\left(1-\frac{r}{n}\right)+(n+r)\log\left(1+\frac{r}{n}\right) > 0,$$

$$\therefore \quad \left(1-\frac{r}{n}\right)^{n-r}.\left(1+\frac{r}{n}\right)^{n+r} > 1, \quad \therefore\ (n-r)^{n-r}.(n+r)^{n+r} > n^{2n},$$

or $$\phi(n-r)\,\phi(n+r) > \{\phi(n)\}^2,$$

whence the stated inequality follows.

[The Arithmetic mean of any number of positive quantities being always greater than the Geometric mean, if we apply this to the case of the n^2 quantities of which $2n-1$ are each $1/(2n-1)$, $2n-3$ each $1/(2n-3)$, &c., we obtain at once the required result.]

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On the Use of / as a Symbol of Operation. *By* Dr. Macfarlane.

It is now the practice of physicists and mathematicians to use the slant bar / as the sign of division in places where such use allows the text to be consolidated. This is the aim which Professor Stokes had in introducing the symbol. But, so far as I know, the great advantages of the new sign, when used as a symbol of operation, have not yet been pointed out.

Viewed as a symbol of operation / denotes *reciprocal*, in the same way as $\surd$ denotes *square root*, and $-$ *reverse*. The expression $/a$ is a sufficient notation for the reciprocal of a. It is not any more necessary to write $1/a$ for the reciprocal of a, than it is to write $0-a$ for $-a$. At page 55 of Vol. xliv., $\frac{1}{Rr}$ is also written $1/Rr$. Now, grant the principle that the laws for the use of the symbol / ought to be made as analogous as possible to those for the other symbols; it will follow that the 1 before the symbol is redundant, and that Rr ought to be enclosed within brackets. When the simple sign / is used to denote *reciprocal*, then it can be detached, and rules for the manipulation of $\times$ and / can be enunciated which are precisely analogous to those for $+$ and $-$.

The fundamental rules of Algebra are then as follows:—

I. Rules of Association.

(1) For $+$ and $-$.

$$+ + = +; \quad + - = -; \quad - + = -; \quad - - = +.$$

(2) For $\times$ and /.

$$\times \times = \times; \quad \times / = /; \quad / \times = /; \quad // = \times.$$

II. Rules of Commutation.

(1) Addition	$a+b = b+a.$
(2) Subtraction	$a-b = -b+a.$
(3) Multiplication	$a \times b = b \times a.$
(4) Division	$a / b = /b \times a.$

III. Rules of Reduction.

(1) Addition	$a+a = 2a.$
(2) Subtraction	$a-a = 0.$
(3) Multiplication	$a \times a = a^2.$
(4) Division	$a / a = 1.$

IV. Rules of Distribution.

(1) Multiplication	$(a+b)(c+d) = ac+ad+bc+bd.$
(2) Division	$(a+b)/c = a/c+b/c.$

The first rule of association explains how $+$ and $-$ are to be used with brackets; and in the same way the second rule explains how $\times$ and / are to be used with brackets. With this notation it is not only possible to write a complex fraction in one line, but it is also easy to get rid of the

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ambiguity which attaches to the common mode of writing. For example, the expression $\frac{2}{\frac{3}{5}}$ is not only awkward to print, it is ambiguous. It may mean either $(2/3)/5$ or $2/(3/5)$, that is, either $2/3/5$ or $2/3\times5$. The ambiguity is ordinarily removed by introducing a difference of size in the figures, or a difference of length in the horizontal bars. It is owing to this unwieldy notation that the laws for the manipulation of fractional expressions are wanting in simplicity. With the above notation and rules, the manipulation is as easy as in the case of an algebraic sum. Thus, to divide $\frac{a}{b}$ by $\frac{c}{d}$,

$$a/b/(c/d) = a/b/c\times d = ad/(bc).$$

It is necessary to put a bracket round the bc, otherwise the meaning would be $a\times d/b\times c$; just as in $a+d-(b+c)$ the bracket is introduced to show that $-$ attaches to both b and c. I have looked over the 44th volume, to find what has been hitherto done as regards this point. In most cases, as in the example already cited, brackets are not introduced, though the sign / is meant to refer to a plurality of factors. Thus, at p. 40,

$$[\Sigma(\cos\alpha)-\cos(\alpha+\beta+\gamma)]^2/\cos\alpha\cos\beta\cos(\alpha+\beta+\gamma),$$

is evidently meant, not the reciprocal of $\cos\alpha$, but the reciprocal of the whole expression which follows. At p. 46, there is

$$(a^2-b^2)/ab = \frac{a}{b}-\frac{b}{a},$$

where the correct view of the matter is clearly indicated. I find, however, that in Mr. TUCKER'S appendix the brackets are introduced; at p. 130, there is $(2\cot\phi-\cot A-\cot B)\,a/(4\cot\phi)$;

at p. 133, there is $2R^2\sigma\sin^2A\sin^2B/(2R\sin A\sin B)$,

and $\sigma bd/(2R)$.

I hold that this practice is the only one which is consistent with the other practices of Algebra. Thus $\sqrt{a}\,b$ means the square root of a, and that multiplied by b, while $\sqrt{(ab)}$ means the square root of ab. Again, ab^2 means a multiplied by b^2, while $(ab)^2$ means the square of ab. While $(mn)!$ means the factorial of mn, $m\,n!$ means m times the factorial of n, and $m!\,n!$ means factorial m multiplied by factorial n. The principle is that the functional sign refers only to the preceding quantity or only to the succeeding quantity, and, if it is to apply to more, the quantities to which it is to apply must be marked off by a bracket.

8528. (By Rev. T. C. SIMMONS, M.A.)—If K be the Lemoine-point of a triangle whose Brocard-points are ω, ω', and whose circumcircle and Brocard-ellipse are denoted respectively by Σ and Σ'; prove that (1) if *any* line be drawn through K, its poles with respect to Σ and Σ' coincide; and hence (2) that Σ and Σ' can be projected into two conics having the projection of K for a common focus and the projection of the polar of K

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for a common directrix, and whose eccentricities are respectively equal to $2\rho'/R$ and ρ'/R. Prove also (3) that, if Σ' revolve round K through an angle of 180°, its common chord with Σ orthogonally trisects the distance between K and the polar of K; (4) that the lines $K\omega$, $K\omega'$ form the diagonals of a quadrilateral inscribed in Σ, and touching Σ' at the extremities of the minor axis; (5) that the chord of Σ through K parallel to $\omega\omega'$ is quadrisected by Σ'.

Solution by R. F. Davis, M.A.; Capt. Brocard; *and others.*

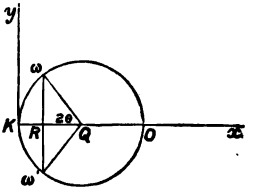
The Brocard-ellipse has its foci at ω, ω', and touches the sides of the triangle ABC; its semi-minor-axis is therefore a mean proportional between the perpendiculars ωm, $\omega' m'$ upon BC. If θ be the Brocard-angle,

$$\omega m = \omega B \sin\theta = a \sin(C-\theta) \sin\theta / \sin C,$$

$$\omega' m' = \omega' B \sin(B-\theta) = c \sin(A-\theta) \sin(B-\theta) / \sin A,$$

and $\sin^3\theta = \sin(A-\theta)\sin(B-\theta)\sin(C-\theta)$, hence $\omega m \cdot \omega' m' = 4R^2 \sin^4\theta$, or semi-minor-axis of $\Sigma' = 2R\sin^2\theta$.

Let O be the circum-centre, Q the middle point of OK; $QO = QK = Q\omega = Q\omega' = \rho'$, the radius of the Brocard-circle; angle ωQK = angle $\omega' QK = 2\theta$. Then the Cartesian equations of Σ, Σ', referred to two rectangular axes KOx, Ky, can be expressed in terms of R, ρ'.



For Σ is the circle centre $(2\rho', 0)$ whose radius is R; its equation is therefore

$$x^2 + y^2 - 4\rho' x - (R^2 - 4\rho'^2) = 0 \quad \text{.....(i.).}$$

Again, Σ' is the ellipse whose foci are ω, ω' and whose semi-minor-axis $= 2R\sin^2\theta$. If $\omega R \omega'$ be the common ordinate,

$$\omega R = \rho' \sin 2\theta \quad \text{and} \quad KR = \rho'(1-\cos 2\theta) = 2\rho' \sin^2\theta.$$

Hence

$$\text{(semi-major-axis)}^2 = \rho'^2 \sin^2 2\theta + 4R^2 \sin^4\theta = 4R^2 \sin^2\theta (\sin^2\theta + \rho'^2 \cos^2\theta / R^2)$$
$$= 4R^2 \sin^2\theta (\sin^2\theta + \cos^2\theta - \tfrac{3}{4}) = R^2 \sin^2\theta.$$

The semi-axes of Σ' are therefore $R\sin\theta$, $2R\sin^2\theta$.

[For if ρ = radius of "T. R. circle," $R^2 = 3\rho^2 + \rho'^2$ and $R = 2\rho\cos\theta$. Whence $\rho'^2 = R^2(1 - \frac{3}{4}\sec^2\theta)$, and θ can be expressed in terms of R, ρ'.]

The coordinates of the centre R of Σ' are $(2\rho'\sin^2\theta, 0)$; and the equation of Σ' is found to be

$$\left.\begin{aligned} \frac{4x^2(R^2-\rho'^2)}{R^2-4\rho'^2} + 4y^2 - 4\rho' x - (R^2 - 4\rho'^2) &= 0 \\ \text{or,} \qquad x^2 \operatorname{cosec}^2\theta + \ldots\ldots\ldots\ldots\ldots\ldots &= 0 \end{aligned}\right\} \quad \text{.....(ii.).}$$

The properties of the question can all be deduced from equations (i.) and (ii.).

1. The polar of the origin K is the same for both curves, namely,

$$-2\rho' x - (R^2 - 4\rho'^2) = 0, \text{ or } x + (R^2 - 4\rho'^2)/2\rho' = 0.$$

If (α, β) be any point on this polar, its polars with respect to Σ, Σ' are respectively $x\alpha + y\beta - 2\rho' x = 0$, and $4x\alpha(R^2-\rho'^2)/(R^2-4\rho'^2) + 4y\beta - 2\rho' x = 0$; and these lines coincide, since $\alpha + (R^2 - 4\rho'^2)/2\rho' = 0$.

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2. If Σ be projected orthogonally so as to become an ellipse with focus K, its eccentricity will $= KO/R = 2\rho'/R$. If Σ' be so projected, its eccentricity will $= KR/$minor axis of $\Sigma' = 2\rho' \sin^2\theta / 2R \sin^2\theta = \rho'/R$. The common directrix will obviously be the projection of the polar of K.

3. If Σ' revolve round K through an angle of 180°, its equation will differ from (ii.) only in the sign of the term $4\rho' x$, and its common chords with Σ are found to be $6\rho' x + R^2 - 4\rho'^2 = 0$ and $2\rho' x + 3(R^2 - 4\rho'^2) = 0$; the first of which orthogonally trisects the distance between K and its polar whose equation is $2\rho' x + R^2 - 4\rho'^2 = 0$.

4. The equations to $K\omega$, $K\omega'$ being $y = \pm x \cot\theta$, the chords of intersection of these lines with Σ are $x^2 \operatorname{cosec}^2\theta - 4\rho' x - (R^2 - 4\rho'^2) = 0$, and coincide with the tangents to Σ' parallel to the axis of y.

5. If $x = 0$; we have for Σ, $y^2 = R^2 - 4\rho'^2$, and for Σ', $4y^2 = \ldots$ Whence, &c.

8576. (By J. J. Walker, F.R.S.)—Show that the axis of the right cone through the (unit) vectors α, β, γ coincides with the vector

$$\Sigma\alpha\,(1 + S\beta\gamma)(1 - S\beta\gamma + S\gamma\alpha + S\alpha\beta).$$

Solution by the Rev. T. R. Terry, M.A.; Prof. Mathews, M.A.; *and others.*

Let ϖ = unit vector along axis $\equiv A\alpha + B\beta + C\gamma$. Then

$$S \cdot \varpi\alpha = S \cdot \varpi\beta = S \cdot \varpi\gamma = \cos\theta,$$

where 2θ is vertical angle of cone. Therefore $-A + B \cdot S \cdot \alpha\beta + C \cdot S \cdot \alpha\gamma$ = two similar expressions. Whence solving A is proportional to

$$(1 + S\beta\gamma)(1 - S\beta\gamma + S\gamma\alpha + S\alpha\beta).$$

8524. (By R. A. Roberts, M.A.)—Show that the boundary of the solar shadow on the plane z of Sir Robert Ball's cylindroid

$$z(x^2 + y^2) - 2mxy = 0$$

is a tricuspidal hypocycloid; and find the coordinates of its centre in terms of the position of the sun.

Solution by Professor Sircom, M.A.

The boundary of the shadow will be the envelope of the projection of a generator whose equations may be written

$$x \cos l + y \sin l = 0, \quad z + 2m \sin l \cos l = 0,$$

on the plane of z by lines parallel to the sun's direction.

Let $\cos\beta \sin\alpha$, $\sin\beta \sin\alpha$, $\cos\alpha$ be the direction cosines of the sun's position. Then, if P be the projection of the point $(0, 0, -2m \sin l \cos l)$, and p the perpendicular from O on the projection of the generator,

$$OP = -2m \sin l \cos l \tan\alpha \quad \text{and} \quad p = 2m \tan\alpha \sin l \cos l \cos(l - \beta),$$

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and the equation of the projection is

$$x\cos l + y\sin l = 2m\tan\alpha\,(\cos\beta\cos^2 l\sin l + \sin\beta\cos l\sin^2 l),$$

or, putting $\cos^3 l = \frac{1}{4}(\cos 3l + 3\cos l)$, and reducing,

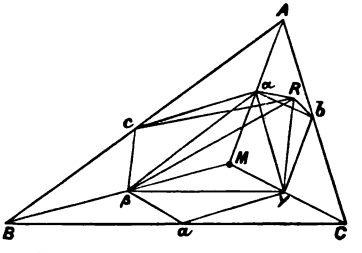
$$(x - \tfrac{1}{2}m\tan\alpha\sin\beta)\cos l + (y - \tfrac{1}{2}m\tan\alpha\cos\beta)\sin l = \tfrac{1}{2}m\tan\alpha\sin(3l-\beta),$$

the envelope of which is a three-cusped hypocycloid with its centre at the point $x = \frac{1}{2}m\tan\alpha\sin\beta$, $y = \frac{1}{2}m\tan\alpha\cos\beta$, the direction of one of its cusps making the angle $\frac{1}{3}\pi + \frac{1}{3}\beta$ with OX, and the others at angles of $\frac{2}{3}\pi$ with this.

8468. (By Professor Bordage.)—A point M is given in the plane of a given triangle ABC. The lines MA, MB, MC are drawn. Prove that the nine-point circles of the triangles MAB, MBC, MCA intersect in a point R lying on the nine-point circle of the triangle ABC. If the point M is on the circum-circle of the triangle ABC, prove that (1) the Simson-lines relative to the triangles MAB, MBC, MCA, ABC and to the respective points C, A, B, M, pass through the point R; (2) the centroids of the four triangles MAB, MBC, MCA, ABC are upon the same circumference.

Solutions by (I.) R. F. Davis, M.A.; (II.) *the* Proposer.

I. Let $a, b, c, \alpha, \beta, \gamma$ be the middle points of BC, CA, AB, MA, MB, MC. Then it can be shewn that the circum-circles of the triangles $a\beta\gamma$, $b\gamma a$, $ca\beta$ intersect on the circum-circle of abc.



For, let the circum-circle $b\gamma a$, $ca\beta$ intersect in R. Then, since

$$\angle bRc = bRa - cRa = \pi - b\gamma a - c\beta a$$
$$= \pi - MAC - MAB$$
$$= \pi - A = \pi - bac,$$

the circum-circle of abc passes through R.

Again,

$$\angle \beta R\gamma = aR\gamma - aR\beta = ab\gamma - \pi + ac\beta = AMC + AMB - \pi = \pi - BMC = \pi - \beta a\gamma,$$

the circum-circle of $a\beta\gamma$ also passes through R.

1. M is on the circum-circle of ABC.

Let H, H_1, H_2, H_3 be the orthocentres of the triangles ABC, MBC, MCA, MAB. Since the triangles ABC, MBC have the same circum-circle, AH is equal and parallel to MH_1; and MH, AH_1 consequently bisect each other. Hence, MH, AH_1, BH_2, CH_3 have all the same middle point R. But the middle point of MH lies on the nine-point circle of ABC, and also on the Simson-line of M with respect to the same triangle. Whence, &c.

2. The nine-point circles of the four triangles have a common point of intersection and a common radius, so that their centres are concyclic. The circum-centre is fixed; hence the four centroids are concyclic.

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II. *Otherwise* :—The nine-point circles of the triangles MAB, MBC pass through the mid-points A_2, B_2, C_1 and A_1, B_2, C_2 of the sides of these triangles. Let R be the second point of intersection of the circumferences. We have : $\angle A_1RC_1 = A_1RB_2 + B_2RC_1 = A_1C_2B_2 + B_2A_2C_1$
$= A_1BB_2 + B_2BC_1 = A_1BC_1 = A_1B_1C_1$.

Since $\angle A_1RC_1$ is equal to $\angle A_1B_1C_1$, the point R is lying on the nine-point circle $A_1B_1C_1$ of the triangle ABC. We could easily prove, by the same manner, that the point R is lying on the nine-point circle of the triangle MAC.

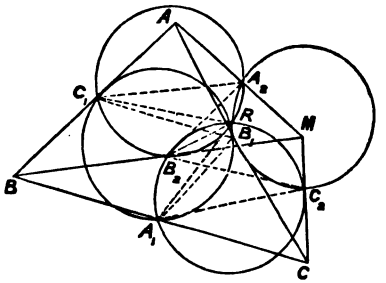

FIG. 1.

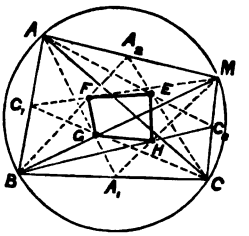

FIG. 2.

(1) When M is on the circum-circle of ABC, the two triangles ABC, MBC having the same basis BC, the two Simson-lines of these triangles, relative to the summits M and A, intersect through the intersection-point R of the nine-point circles of the same triangles. (2) Let E, F, G, H be the centroids (determined by the intersection of two medians) of the four triangles MAC, MAB, ABC, MBC. We have $\frac{A_2F}{A_2B} = \frac{1}{3} = \frac{A_2E}{A_2C}$, and then FE is parallel to BC. For the same motive FG, GH, HE are parallel to CM, MA, AB and equal to the *third* ($\frac{1}{3}$) of these lines. The quadrilateral EFGH is similar to the quadrilateral BCMA and inscriptible. Then the four points E, F, G, H are *concyclic*.

8499. (By EMILE VIGARIÉ.)—Dans un triangle ABC on mène des parallèles à la base BC. Démontrer que les diagonales des trapèzes ainsi formés se coupent sur la médiane AM du triangle donné.

Solution by Rev. T. R. TERRY, M.A. ; Prof. IGNACIO BEYENS ; *and others.*

Taking AB, AC as axes, AB = a, AC = b, the equations to the diagonals of a trapezium are $\frac{x}{\lambda a} + \frac{y}{\mu b} = 1$, $\frac{x}{\mu a} + \frac{y}{\lambda b} = 1$; and where they meet we have $\frac{x}{a} = \frac{y}{b}$, that is to say they meet on the line joining A to the mid-point of BC.

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8339. (By Rev. T. R. Terry, M.A.)—If $\phi(p, q)$ be a function of p and q defined by the equations

$$(2q-2p-1)\phi(p, q) = \phi(p, q-1) = -(p+1)\phi(p+1, q),$$

and if $\phi(1, 0) = 1$, find the value of $\phi(p, q)$.

Solution by H. London, B.A.; Professor Sircom; *and others.*

$$\phi(1.0) = 1, \quad \phi(1.1) = \frac{\phi(1.0)}{-1} = -1, \quad \phi(1.2) = \frac{\phi(1.1)}{1} = -1,$$

$$\phi(1.3) = \frac{\phi(1.2)}{3} = -\frac{1}{3}, \quad \phi(1.4) = \frac{\phi(1.3)}{5} = -\frac{1}{3.5},$$

$$\phi(1.q) = -\frac{1}{1.3.5\ldots(2q-3)}, \quad \phi(2.q) = -\frac{2q-3}{2}\phi(1.q),$$

$$\phi(3.q) = -\frac{2q-5}{3}\phi(2.q),$$

$$\phi(p.q) = (-1)^{p-1}\frac{(2q-3).(2q-5)\ \ldots\ (2q-2p+1)}{p!}\phi(1-q).$$

$$\phi(p.q) = (-1)^p\frac{(2q-3)(2q-5)\ldots(2q-2p+1)}{1.3.5\ldots(2q-3)\,p!} = (-1)^p\frac{2q-p\,(q-p)!}{p![2(q-p)!]}.$$

[Multiplying the equations by $(-1)^p.1.3.5\ldots(2q-2p-3).p!$ and putting $(-1)^p.1.3.5\ldots(2q-2p-3)(2q-2p-1)p!\,Q(pq) = F(pq)$, we shall have $F(pq) = F(p, q-1) = F(p+1, q) =$ constant; therefore

$$F(pq) = F(1, 0) = Q(1, 0) = 1,$$

and
$$Q(pq) = \frac{(-1)^p}{1.3.5\ldots(2q-2p-1).p!}\Big].$$

8426. (By Professor Mahendra Nath Ray, M.A., LL.B.)—Show that the series

$$\tan x + \tfrac{1}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x + \ldots \text{ to } \infty = \tfrac{1}{2}\sqrt{(-1)}\sec^{-1}(\cos 2x).$$

Solution by Prof. Tanner, M.A.; Rev. T. R. Terry, M.A.; *and others.*

When $1 > \tan x > -1$,

$$s = \tan x + \tfrac{1}{3}(\tan x)^3 + \tfrac{1}{5}(\tan x)^5 + \ldots = \tanh^{-1}(\tan x),$$

therefore
$$\tan x = \tanh s = -i\tan is.$$

Hence $\cos 2x = \sec 2is$, which is the form required. We may also write

$$s = \tfrac{1}{2}\tanh^{-1}(\sin 2x) = \frac{1}{2i}\,\mathrm{gd}\,2ix = -\tfrac{1}{2}\mathrm{gd}^{-1}\,2x.$$

[For notation see Greenhill's *Calculus* or Lock's *Higher Trigonometry*. Denoting the series by s, we have $si = \tan^{-1}(i\tan x)$; wherefore $\sec 2si = (1+i^2\tan^2 x)/(1-i^2\tan^2 x) = \cos 2x$, &c.

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Otherwise : — The series $= i \tan^{-1}(-i \tan x) = \frac{1}{2} i \tan^{-1}(-i \sin 2x)$ $= \frac{1}{2} i \sec^{-1}(\cos 2x)$. The more usual form for the sum of this series is $\tan h^{-1}(\tan x)$].

8280. (By Professor SÁRADÁRANJAN RÂY, M.A.)—If ρ be the length of a normal from (α, β) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, prove that

$$\frac{\rho^2}{k^2} = \frac{\alpha^2}{(k+a^2)^2} + \frac{\beta^2}{(k+b^2)^2},$$ where k is given by $\frac{a^2\alpha^2}{(k+a^2)^2} + \frac{b^2\beta^2}{(k+b^2)^2} = 1$.

Solution by Prof. IGNACIO BEYENS ; R. KNOWLES, B.A. ; *and others.*

The equation to a normal at (x_1, y_1) through α, β is $\beta - y_1 = \frac{a^2 y_1}{b^2 x_1}(\alpha - x_1)$

and $$\rho^2 = (\beta - y_1)^2 + (\alpha - x_1)^2 = \left(\frac{a^4 y_1^2}{b^4 x_1^2} + 1\right)(\alpha - x_1)^2 \quad \ldots\ldots\ldots\ (1),$$

and, substituting $x_1 = \frac{a^2\alpha}{(k+a^2)}$, $y_1 = \frac{b^2\beta}{(k+b^2)}$ in (1), and the equation to the ellipse, we obtain the two equations in the question.

8439. (By Professor CLARKE, F.R.S.)—A fine string attached to a fixed point A carries a small ring of mass m, and, passing over a small pulley B in the same horizontal plane with the fixed point, has a mass $(m_1 + m_2)$ attached to the free extremity. The system being in equilibrium, the mass m_2 is removed ; show that the strain on the fixed point is instantly reduced by $\frac{m_2(m_1+m_2)}{(m_1+m_2)^2 + mm_1}$ of its former value.

Solution by G. HEPPEL, M.A. ; Rev. T. GALLIERS, M.A. ; *and others.*

Let α be the angle the string makes with horizontal when there is equilibrium ; T the tension after the removal of m_2 ; f and f_1 the initial accelerations of m and m_1. Then, from equilibrium $2(m_1+m_2)\sin\alpha = m$; and from virtual work $m_1 + m_2 : m = f : f_1$; finally, $2T \sin\alpha = m(g - f)$; $T = m_1(g + f_1)$. From these four equations, α, f, f_1 may be eliminated, giving $T = (m_1+m_2)g \cdot \frac{m_1(m+m_1+m_2)}{(m_1+m_2)^2 + mm_1}$. Hence, since the original tension was $(m_1+m_2)g$, it has been reduced by a fraction of its former value equal to $\frac{m_2(m_1+m_2)}{(m_1+m_2)^2 + mm_1}$.

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Notes on Questions 7954, 7624, 8307. *By the* Editor.

We collect together here some corrections or improvements of solutions that have already appeared in our volumes.

(7954.) As the solution given on p. 71, of Vol. xliii., involves a knowledge of the values of r, r_1, and is moreover incomplete, we add here the Proposer's Solution :—

Let O, O_1 be the in-centre and ex-centre, and E the point where AOO_1 cuts the base; then, by inspection from a figure in either the triangle or the tetrahedron, we have

$$\frac{AO}{r} = \frac{AO_1}{r_1} \text{ and } \frac{r}{OE} = \frac{r_1}{O_1E};$$

therefore $$\frac{AO}{OE} = \frac{AO_1}{O_1E} \text{ and } \frac{AO}{OE} = \frac{p_1 - r}{r} \text{ and } \frac{AO_1}{O_1E} = \frac{p_1 + r_1}{r_1};$$

therefore $$\frac{p_1}{r} - 1 = \frac{p_1}{r_1} + 1 \text{ or } \frac{1}{r} - \frac{1}{r_1} = \frac{2}{p_1}.$$

(7624.) In reply to Mr. Simmons, who, on page 31 of Volume xlv., expresses his belief that the result obtained in the first solution of this question is incorrect, Mr. Biddle says, that "the easy applicability of a result is no guarantee of its accuracy. No doubt the result given by Mr. Simmons is not only simple and easily applicable, but also correct in two possible cases, viz., if A be the only passenger in the train when B enters it, or if on the other hand there be only one vacant place for B to take. In these two cases, the chance of their travelling in the same compartment is clearly $(q-1)/(pq-1)$. But, in the intermediate cases, the chance varies, being greater than at the extremes; and it is misleading to assert, as having any bearing upon the question, that 'A being supposed already seated, the chance that any subsequent passenger will occupy any particular seat is clearly the same whether he enter the train first or last of the remaining $pq-m$ men.' There is a great difference between the chances that B will travel with A, according as he is (1) one of $pq-m$ men getting into the train after A, or (2) an individual joining the $pq-m$ men (of whom A is one) already seated. I have nothing in my solution to recall."

To this Mr. Simmons rejoins as follows :—"If Mr. Biddle is not to be convinced by abstract reasoning, let us take the particular instance of 3 compartments, each of 3 seats, and test the numerical results.

(i.) Suppose that, besides A, *two* other passengers, P and Q, are already seated. Here the respective chances that P and Q are *both*, *one only*, and *neither* with A are $\frac{1.2}{8.7}$ or $\frac{1}{28}$, $2.\frac{2}{8}.\frac{6}{7}$ or $\frac{3}{7}$, and $\frac{6}{8}.\frac{5}{7}$ or $\frac{15}{28}$. A new passenger B now being supposed to enter, the chance of his companying with A is 0 in the first case, $\frac{1}{6}$ in the second, and $\frac{1}{3}$ in the third. Hence the whole chance, for *two* passengers P and Q, that B companies with A is $\frac{1}{28}.0 + \frac{3}{7}.\frac{1}{6} + \frac{15}{28}.\frac{1}{3} = \frac{1}{14} + \frac{5}{28} = \frac{1}{4}$.

(ii.) Suppose that, besides A, there are already *three* other passengers, P, Q, and R. Here the respective chances that *all*, *two*, *one*, and *none* of them company with A are 0, $3.\frac{1}{4}.\frac{1}{7}$ or $\frac{3}{28}$, $3.\frac{1}{4}.\frac{6.5}{7.6}$ or $\frac{15}{28}$, and $\frac{6.5.4}{8.7.6}$

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or $\frac{5}{14}$. The several chances of B companying with A under these four suppositions are 0, 0, $\frac{1}{2}$, and $\frac{2}{3}$. So that for *three* passengers, P, Q, R, the whole chance for B is

$$0 \cdot 0 + \tfrac{3}{12} \cdot 0 + \tfrac{15}{28} \cdot \tfrac{1}{2} + \tfrac{5}{14} \cdot \tfrac{2}{3} = \tfrac{3}{12} + \tfrac{1}{7} = \tfrac{1}{2}.$$

(iii.) Moreover, if there had only been *one* passenger P besides A, the chance for B would have been $\frac{2}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{2}$; and if B and A had been the *only* passengers it would have been $\frac{2}{3} = \frac{1}{2}$.

Thus, whether there are altogether five, four, three, or two passengers in the whole train, the chance of B companying with A is in each case the same, namely $\frac{1}{2}$. And this being in agreement with the formula $(q-1)/(pq-1)$, my belief in the correctness of the very simple solution on p. 31 of Vol. XLV. is now strengthened into positive conviction.

I am unable to see wherein the above numerical results are wrong, or how it can be shown, by other numerical instances, that the formula $(q-1)/(pq-1)$ is incorrect. Mr. BIDDLE's method transforms a molehill into a mountain, and is so difficult that all my efforts to understand it have hitherto been in vain."

(8307.) The solution by exponentials given on p. 24, of Vol. XLV., may be well supplemented by the following neat proof by Mr. R. F. DAVIS:—

Since $$\sin(n+1)x = \sin x \cos nx + \cos x \sin nx,$$

$$\sin(n+1)x \operatorname{cosec} x = \cos nx + \cos x \sin nx \operatorname{cosec} x.$$

Similarly $$\sin nx \operatorname{cosec} x = \cos(n-1)x + \cos x \sin(n-1)x \operatorname{cosec} x,$$

$$\sin(n-1)x \operatorname{cosec} x = \cos(n-2)x + \cos x \sin(n-2)x \operatorname{cosec} x.$$

Hence, substituting, we have

$$\sin(n+1)x \operatorname{cosec} x = \cos nx + \cos x \cos(n-1)x + \cos^2 x \cos(n-2)x + \ldots + \cos^n x.$$

8269. (By ASPARAGUS.)—From a point O on a parabola there are drawn chords OP, OQ normal to the parabola at P, Q; prove that (1) cos POQ = 3 cos θ, where θ is the angle which the normal at O makes with the axis; and (2) OP × OQ = latus rectum × diameter of curvature at O.

Solution by MAHENDRA RAY, M.A.; R. KNOWLES, B.A.; *and others.*

The equation to any normal to the parabola $y^2 = 4ax$ is of the form $y = mx - 2am - am^3$, where m is the tangent of the angle which the normal makes with the axis.

Therefore, if x' and y' be the coordinates of O, the tangents of the angles which the normal at O and the straight lines OP and OQ respectively make with the axis are the three roots of the cubic equation

$$am^3 + (2a - x')m + y' = 0.$$

Let the roots be m_1, m_2, m_3, so that $m_1 = \tan\theta = -\dfrac{y'}{2a}$, from the known direction of the normal at O. Also, we shall have

$$m_1 + m_2 + m_3 = 0, \quad m_1 m_2 m_3 = -\frac{y'}{a} = 2m_1,$$

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so that $m_2 + m_3 = -m_1 = \frac{y'}{2a}$, $m_2 m_3 = 2$..... (1, 2).

Therefore $\tan \mathrm{POQ} = \frac{m_2 - m_3}{1 + m_2 m_3} = \frac{(m_1^2 - 8)^{\frac{1}{2}}}{3}$, from equations (1) and (2).

Therefore $\sec^2 \mathrm{POQ} = 1 + \tan^2 \mathrm{POQ} = \frac{1}{9}(1 + m_1^2) = \frac{1}{9}\sec^2\theta$,

or $\cos \mathrm{POQ} = 3\cos\theta$.

Again, it may be easily shown that, if the straight line $y = mx + c$ intersect the parabola $y^2 = 4ax$, the length of the chord intercepted is $\frac{4\sqrt{\{(a^2 - mac)(1 + m^2)\}}}{m^2}$; but the equation to OP is $y = m_2 x + y' - m_2 x'$,

therefore $\mathrm{OP} = \frac{2\sqrt{\{(4a^2 - 4am_2 y' + 4ax' m_2^2)(1 + m_2^2)\}}}{m_2^2}$

$= \frac{2(2a - m_2 y')\sqrt{(1 + m_2^2)}}{m_2^2}$, since $y'^2 = 4ax'$.

Similarly $\mathrm{OQ} = \frac{2(2a - m_3 y')\sqrt{(1 + m_3^2)}}{m_3^2}$;

therefore $\mathrm{OP}.\mathrm{OQ} = \frac{4(2a - m_2 y')(2a - m_3 y')\sqrt{\{(1 + m_2^2)(1 + m_3^2)\}}}{m_2^2 m_3^2}$

$= \frac{4(4a^2 + y'^2)\left(1 + \frac{y'^2}{4a^2}\right)^{\frac{1}{2}}}{m_2^2 m_3^2}$, from equations (1) and (2),

$= \frac{(4a^2 + y'^2)^{\frac{3}{2}}}{2a} = \frac{(4a)^{\frac{3}{2}}(a + x')^{\frac{3}{2}}}{2a} = \frac{4a(a + x')^2}{[a(a + x')]^{\frac{1}{2}}} = \frac{4\mathrm{SA}.\mathrm{SO}^2}{(\mathrm{SA}.\mathrm{SO})^{\frac{1}{2}}} = \frac{4\mathrm{SA}.\mathrm{SO}^2}{\mathrm{SY}}$,

(where SY is the perpendicular to the tangent at O)

= latus rectum × diameter of curvature at O.

8462. (By L. J. Rogers, B.A.)—If (1) $u + v + w = \mathrm{K}$, show that

$$\mathrm{Z}u + \mathrm{Z}v + \mathrm{Z}w = \frac{k^2}{k'} \operatorname{cn} u \operatorname{cn} v \operatorname{cn} w;$$

and if (2) $u + v + w = \mathrm{K} + i\mathrm{K}'$, then

$$\mathrm{Z}u + \mathrm{Z}v + \mathrm{Z}w + \frac{i\pi}{2\mathrm{K}} = \frac{i}{k'} \operatorname{dn} u \operatorname{dn} v \operatorname{dn} w.$$

Solution by J. Hammond, M.A.

By means of the addition formula

$$\mathrm{Z}u + \mathrm{Z}v = \mathrm{Z}(u + v) + k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn}(u + v),$$

we immediately obtain

$$\mathrm{Z}u + \mathrm{Z}v + \mathrm{Z}w = \mathrm{Z}(u + v + w) + k^2 \operatorname{sn}(u + v)\{\operatorname{sn} u \operatorname{sn} v + \operatorname{sn} w \operatorname{sn}(u + v + w)\}$$

.....(a).

And in the first case (where $u + v + w = \mathrm{K}$) this formula is simplified by

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writing Z (K) = 0 and sn K = 1, when it becomes

$$Zu + Zv + Zw = k^2 \operatorname{sn}(u+v)\left\{\operatorname{sn} u \operatorname{sn} v + \operatorname{sn}(K-u-v)\right\}$$

$$= k^2 \operatorname{sn}(u+v)\left\{\operatorname{sn} u \operatorname{sn} v + \frac{\operatorname{cn}(u+v)}{\operatorname{dn}(u+v)}\right\}$$

$$= \frac{k^2 \operatorname{sn}(u+v)}{\operatorname{dn}(u+v)}\left\{\operatorname{sn} u \operatorname{sn} v \operatorname{dn}(u+v) + \operatorname{cn}(u+v)\right\}$$

$$= \frac{k^2 \operatorname{sn}(u+v)}{\operatorname{dn}(u+v)} \operatorname{cn} u \operatorname{cn} v = \frac{k^2 \operatorname{sn}(K-w)}{\operatorname{dn}(K-w)} \operatorname{cn} u \operatorname{cn} v = \frac{k^2}{k'} \operatorname{cn} u \operatorname{cn} v \operatorname{cn} w.$$

In the second case (where $u+v+w = K+iK'$) a precisely similar simplification of (*a*) is effected by writing

$$Z(K+iK') = -\frac{i\pi}{2K} \text{ and } \operatorname{sn}(K+iK') = \frac{1}{k},$$

when it becomes

$$Zu + Zv + Zw + \frac{i\pi}{2K}$$

$$= k^2 \operatorname{sn}(u+v)\left\{\operatorname{sn} u \operatorname{sn} v + \frac{1}{k} \operatorname{sn}(K+iK'-u-v)\right\}$$

$$= k^2 \operatorname{sn}(u+v)\left\{\operatorname{sn} u \operatorname{sn} v + \frac{\operatorname{dn}(u+v)}{k^2 \operatorname{cn}(u+v)}\right\}$$

$$= \frac{\operatorname{sn}(u+v)}{\operatorname{cn}(u+v)}\left\{k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{cn}(u+v) + \operatorname{dn}(u+v)\right\}$$

$$= \frac{\operatorname{sn}(u+v)}{\operatorname{cn}(u+v)} \operatorname{dn} u \operatorname{dn} v = \frac{\operatorname{sn}(K+iK'-w)}{\operatorname{cn}(K+iK'-w)} \operatorname{dn} u \operatorname{dn} v$$

$$= \frac{\operatorname{dn} w}{-ik'} \operatorname{dn} u \operatorname{dn} v = \frac{i}{k'} \operatorname{dn} u \operatorname{dn} v \operatorname{dn} w.$$

8449. (By R. Tucker, M.A.)—Draw the Lemoine-lines DKE′, D′KF, EKF′, *i.e.*, parallels to the sides, through the Lemoine-point (K) of a triangle, and show that (1) K is the centroid of the Lemoine-points of the triangles BD′F, CDE′, AF′E; (2) if Ω, Ω′ are the Brocard-points, and ρ_1, ρ_1', &c. the circum-radii of BΩC, BΩ′C, &c., then $\rho_1\rho_2\rho_3 = R^3 = \rho_1'\rho_2'\rho_3'$; (3) if A′B′C′ is the first Brocard-triangle (see Casey's *Conics*), AA′, BB′, CC′ intersect on Kiepert's hyperbola (*cf.* Casey), where it is met by Pπ (see Quest. 7819, P being the orthocentre); (4) the triangle formed with sides equal to the median lines of ABC has the same Brocard-angle as ABC, and, if A_1, B_1, C_1 are the angles of this triangle,

$$\cot A + \cot A_1 = \cot B + \cot B_1 = \cot C + \cot C_1 = \tfrac{2}{3}\cot\omega,$$

and its Brocard-radius is to the Brocard-radius of ABC as $m_1m_2m_3$ is to $3abc$, where m_1, m_2, m_3 are the aforesaid median lines; (5) if A″B″C″ be the second Brocard-triangle, Δ″ its area, and ρ_1', R the Brocard and circum-radius respectively of the triangle formed by the median lines as above, $\Delta''R_1^4/\Delta^2\rho_1'^2 \propto \cot^3\omega$.

Solution by Asûtosh Mukhopâdhyây, M.A., F.R.A.S.

(1) By Quest. 8167, a mass *m* placed at the Lemoine-point of the triangle BD′F is equivalent to masses $\frac{a^2}{k}.m$, $\frac{b^2}{k}.m$, and $\frac{c^2}{k}.m$ placed

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respectively at the points F, B, and D′ (where $k = a^2+b^2+c^2$). But mass a^2m/k at F is equivalent to $a^2(a^2+c^2)\,m/k^2$ at A and a^2b^2m/k^2 at B; and so on for the other mass. Hence, mass m at the Lemoine-point of BD′F is equivalent to masses

$$a^2(a^2+c^2)\,m/k^2,\quad b^2(a^2+c^2+k)\,m/k^2,\quad \text{and}\quad c^2(a^2+c^2)\,m/k^2,$$

placed respectively at the points A, B, C. Similarly for the Lemoine-points of the triangles CDE′ and AF′E. Hence the centroid of equal masses m placed at the three Lemoine-points is the same as those of $3a^2m/k$, $3b^2m/k$, and $3c^2m/k$ placed at the angular points A, B, C, and this, by Question 8167, is the Lemoine-point of the triangle ABC.

(2) $$\rho_1 = \mathrm{BC}/2\sin \mathrm{B\Omega C} = a/2\sin \mathrm{C} = \frac{a}{c}.\,\mathrm{R}.$$

Similarly $\rho_2 = \frac{b}{a}.\,\mathrm{R}$ and $\rho_3 = \frac{c}{b}.\,\mathrm{R}$, $\therefore\ \rho_1\rho_2\rho_3 = \mathrm{R}^3 = \rho_1'\rho_2'\rho_3'$.

(3) A′ is the intersection of BΩ with the Brocard-circle, *i.e.*, of

$$b^2\alpha - ac\gamma = 0 \text{ with } abc(a^2+\beta^2+\gamma^2) = a^3\beta\gamma + b^3\gamma\alpha + c^3\alpha\beta.$$

Eliminating α between these equations, we have $(b^3\beta - c^3\gamma)(bc\beta - a^2\gamma) = 0$; but $bc\beta - a^2\gamma = 0$ is the equation to AΩ, therefore $b^3\beta - c^3\gamma = 0$ represents AA′. Similarly $c^3\gamma - a^3\alpha = 0$ and $a^3\alpha - b^3\beta = 0$ represent the lines BB′ and CC′ which evidently meet at the point $a^3\alpha = b^3\beta = c^3\gamma$ which lies both on $\alpha\beta\sin(\mathrm{A}-\mathrm{B}) + \beta\gamma\sin(\mathrm{B}-\mathrm{C}) + \gamma\alpha\sin(\mathrm{C}-\mathrm{A}) = 0$ (Kiepert's Hyperbola) [for $\Sigma\sin^3\mathrm{C}\sin(\mathrm{A}-\mathrm{B}) = \Sigma\sin^2\mathrm{C}(\sin^2\mathrm{A}-\sin^2\mathrm{B}) = 0$],

and on the line $\mathrm{P}\pi$ whose equation is

$$a^3\alpha\cos\mathrm{A}\sin(\mathrm{B}-\mathrm{C}) + b^3\beta\cos\mathrm{B}\sin(\mathrm{C}-\mathrm{A}) + c^3\gamma\cos\mathrm{C}\sin(\mathrm{A}-\mathrm{B}) = 0.$$

(4) $4m_1^2 = 2b^2+2c^2-a^2$, &c., therefore

Δ_1 = area of triangle whose sides are the median lines

$$= \tfrac{1}{4}\left[4m_1^2m_2^2 - (m_1^2+m_2^2-m_3^2)^2\right]^{\frac{1}{2}} = \tfrac{3}{16}\left(2a^2b^2+\ldots+\ldots-a^4-\ldots-\ldots\right) = \tfrac{3}{4}\Delta^{\frac{1}{2}};$$

$$\cot\omega = \frac{a^2+b^2+c^2}{4\Delta} \text{ and } \cot\mathrm{A} = \frac{b^2+c^2-a^2}{4\Delta},$$

$$\cot\mathrm{A}_1 = \frac{m_2^2+m_3^2-m_1^2}{4\Delta_1} = \frac{5a^2-b^2-c^2}{12\Delta};$$

$$\therefore\quad \cot\mathrm{A}+\cot\mathrm{A}_1 = (2b^2+2c^2+2a^2)/12\Delta = (a^2+b^2+c^2)\,6\Delta$$
$$= \tfrac{3}{2}\cot\omega = \cot\mathrm{B}+\cot\mathrm{B}_1 = \cot\mathrm{C}+\cot\mathrm{C}_1.$$

$\frac{m_1^2+m_2^2+m_3^2}{4\Delta_1} = \frac{a^2+b^2+c^2}{4\Delta}$. Hence the Brocard-angles of the two triangles are equal.

The Brocard radius $= \frac{\mathrm{R}}{2}(1-3\tan^2\omega)^{\frac{1}{2}} = \left(\frac{abc}{8\Delta}\right)^{\frac{1}{2}}$. Since the two triangles have the same Brocard-angle, the Brocard radii are to one another as $\frac{abc}{\Delta} : \frac{m_1m_2m_3}{\Delta_1}$ or $3abc : 4m_1m_2m_3$.

(5) Let the intersection of AKA″ with the circle $abc(a^2+\beta^2+\gamma^2) = a^3\beta\gamma + b^3\gamma\alpha + c^3\alpha\beta$ be given by the equations $\frac{\alpha}{m} = \frac{\beta}{b} = \frac{\gamma}{c}$. Then

$am^2 - (b^2+c^2)\,m + a(b^2+c^2-a^2) = 0$, or $m = a$ or $b^2+c^2-a^2/a = 2bc\cos\mathrm{A}/a$.

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$m = a$ corresponds to the point k. Therefore the coordinates A″ B″ are $(2bc\cos A/a)\,k_1$, bk_1, ck_1, where $k_1 = \Delta/m_1^2$. Similarly for and of C″,

therefore $$\Delta'' = -\frac{1}{8\Delta^2}\cdot\frac{\Delta^3}{m_1^2m_2^2m_3^2}\begin{vmatrix} 2bc\cos A, & ab, & ac \\ ab, & 2ca\cos B, & bc \\ ac, & bc, & 2ab\cos B \end{vmatrix};$$

or $$\Delta'' = -\frac{\Delta}{8m_1^2m_2^2m_3^2}(3a^2b^2c^2 - a^6 - b^6 - c^6).$$

But $$m_1m_2m_3 = 4\Delta_1R_1 = 3\Delta R_1,$$

$$3a^2b^2c^2 - a^6 - b^6 - c^6 = 4\Delta\cot\omega\,(12\Delta^2 - 4\Delta^2\cot^2\omega) = 16\Delta^3\cot^3\omega\,(3\tan^2\omega - 1),$$

and $\rho_1^2 = \frac{1}{4}R_1^2(1 - 3\tan^2\omega)$, $\therefore$ $\Delta''\,.\,9\Delta^2R_1^2 = 16\Delta^4\cot^3\omega\,(1 - 3\tan^2\omega)$,

or $\Delta''\,.\,9R_1^4 = 64\Delta^2\rho_1^2\cot^3\omega$ or $\Delta''R_1^4/\Delta^2\rho_1^2$ varies as $\cot^3\omega$.

8365. (By W. J. C. Sharp, M.A.)—Prove that (1) every unicursal twisted quartic curve lies on a quadric surface; (2) every unicursal twisted sextic on a cubic surface; and (3) every unicursal twisted octavic on two quartic surfaces.

Solution by Professors Sircom, M.A., *and* Ignacio Beyens.

Let $x = a_1\theta^4 + a_2\theta^3 + a_3\theta^2 + a_4\theta + a_5$, so for y, z, be the equations of a unicursal twisted quartic.

A quadric in xyz has nine independent constants; if then the above values of x, y, z be substituted in the equation of the quadric surface, we have $A\theta^8 + A_1\theta^7 + \ldots = 0$, and the curve lies on the surface if each coefficient A is 0. Hence we have nine simple equations $A = 0$, to determine the nine constants of the quadric.

Similarly a cubic contains 19 disposable constants, and they can be determined by substituting $x = a_1\theta^6 + \ldots$, so for y and z, giving an equation $Ax^{18} + A_1x^{17} + \ldots = 0$ containing 19 terms; then the equations $A = 0$ determine the coefficients of the cubic on which the sextic lies.

A quartic contains 34 disposable constants; substituting the expressions for xyz of a twisted octavic, we have $A\theta^{32} + A_1\theta^{31} + \ldots = 0$, containing 33 terms; whence 33 of the disposable constants can be expressed in terms of the 34th and known quantities, or the quartic will be of the form $aU + U_1 = 0$, and the octavic lies on the two quartics $U = 0$, $U_1 = 0$.

8295. (By Belle Easton.)—Three bodies A, B, and C, attracting each other according to Newton's law, continue during their motion in a straight line with each other; show that, if C is the middle body,

$$Aa^2(b^3 - c^3) + Bb^2(c^3 - a^3) = Cc^2(a^3 - b^3),$$

where $a = BC$, $b = CA$, $c = AB$, all taken positively.

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Solution by D. EDWARDES; Professor NASH, M.A.; *and others.*

This case of the problem of three bodies is discussed in LIOUVILLE's *Journal de Mathématiques*, Tom. i., 2nd Series. If x, y be the coordinates of B, and x', y' of C, then we have $x = (1+p)\,x'$, $y = (1+p)\,y'$, where p is the positive root of

$$\frac{A+B}{(1+p)^3} + \frac{C}{1+p} + \frac{C}{p^2(1+p)} = A + C + \frac{B}{(1+p)^2} - \frac{B}{p^2};$$

but, from the above equations, $a = pb$, $c = (1+p)\,b$; whence, substituting and reducing, we have the stated result.

8369. (By Rev. J. J. MILNE, M.A. Connected with Professor CROFTON's Question 8310.)—An event happens on an average once a year. Show that the chance that it will not happen in any particular year is $1/\epsilon$.

Solution by the PROPOSER.

If n be very great, n events happen in n years, and the chance that any particular one of the events does not happen in the given year is $\frac{n-1}{n}$. Similarly, the chance that any other particular event does not happen in the given year is $\frac{n-1}{n}$, therefore $\left(\frac{n-1}{n}\right)^2$ is the chance that neither of these events happens in the given year.

Similarly the chance that no one of the events happens in the given year

$$= \left(\frac{n-1}{n}\right)^n = \left(1-\frac{1}{n}\right)^n = 1 - \frac{n}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \ldots$$

$$= 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots \text{ when } n \text{ is very great, } = \epsilon^{-1}.$$

8361. (By Capt. H. BROCARD. Suite au Question 8326.)—Trouver l'expression des distances des points d'intersection précités aux symédianes.

Solution by the PROPOSER.

Le cercle décrit sur GH comme diamètre a pour équation

$$\Sigma a^2 \sin 2A - \Sigma a\beta \sin C = 0$$

(CASEY's, *Analytical Geometry*, p. 120). Les intersections de ce cercle avec la médiane $b\beta = c\gamma$ ont pour coordonnées

$$2a \cot A = \frac{\beta}{\sin C} = \frac{\gamma}{\sin B},$$

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et, en tenant compte de la relation $a\alpha + b\beta + c\gamma = 2\Delta$, il vient

$$\alpha = K\frac{a}{m^2-2a^2}, \quad \beta = \frac{K}{b}, \quad \gamma = \frac{K}{c},$$

avec $$K = \frac{2\Delta\,(m^2-2a^2)}{2m^2-3a^2}, \quad m^2 = a^2+b^2+c^2.$$

La distance de ce point à la symédiane $\frac{\beta}{b} - \frac{\gamma}{c} = 0$ a donc pour expression, d'après la formule classique (*loc. cit.*, p. 55), $p_a = K\frac{c^2-b^2}{abc}$. On en conclut, pour les distances cherchées $p_a = \frac{(m^2-2a^2)(c^2-b^2)}{2R\,(2m^2-3a^2)}$, et deux autres formules analogues.

Une de ces distances est nulle dans le cas du triangle rectangle et du triangle isoscèle, comme on pouvait le prévoir.

8226. (By R. Knowles, B.A.)—An ordinate from a focus of an ellipse is produced to meet the auxiliary circle in T; tangents TP, TQ meet the ellipse in P and Q; prove that the radius of the circle TPQ : chord PQ = $(1+3e^2)^{\frac{1}{2}} : 4e$, e being the eccentricity of the ellipse.

Solution by R. J. Brevor, B.A.; A. Gordon; *and others.*

$\sin PTQ = \sin STS' = \frac{2ae}{(b^2+4a^2e^2)^{\frac{1}{2}}} = \frac{2ae}{(1+3e^2)^{\frac{1}{2}}}$; hence the result.

[The point P coincides with the vertex of the minor axis.]

8212. (By D. Edwardes.)—If SY, S'Y' are the focal perpendiculars on a tangent through a point O to an ellipse; SZ, S'Z' the focal perpendiculars on the other tangent through O: prove that (1) YZ, Y'Z', and the straight line through O perpendicular to the major axis, are concurrent; and (2), if YZ meet OS' in L, Y'Z' meet OS in M, the points L, M, S, S' are concyclic.

Solution by G. G. Morrice, B.A.; A. Gordon; *and others.*

1. Let YZ, Y'Z' meet in T; draw circles through OYSZ, OY'S'Z', and YY'Z'Z; and let the circles OYSZ, OY'S'Z' cut OT in K and K' respectively. Then TK . TO = TZ . TY = TZ'. TY' = TK'. TO, therefore K' coincides with K. Since OKS, OKS' are right angles, K, S, S' are in one straight line (viz., the axis) which is perpendicular to OT.

2. OM, OL are perpendicular to Y'Z', YZ respectively; and, since circles pass through YY'S'L, YY'MS, OL . OS' = OY . OY' = OS . OM, therefore L, S', M, S lie on a circle.

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7840. (By Professor WOLSTENHOLME, M.A., Sc.D.)—If θ be any angle between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, prove that (1) for all real values of n

$$\frac{\sin n\theta}{\cos\theta} = n\sin\theta + n(2^2-n^2)\frac{\sin^3\theta}{3!} + n(2^2-n^2)(4^2-n^2)\frac{\sin^5\theta}{5!} + \ldots,$$

$$\frac{\cos n\theta}{\cos\theta} = 1 + (1^2-n^2)\frac{\sin^2\theta}{2!} + (1^2-n^2)(3^2-n^2)\frac{\sin^4\theta}{4!} + \ldots;$$

(2) if $x = \sin\theta$, and $y = \dfrac{\sin n\theta}{\cos\theta}$ or $\dfrac{\cos n\theta}{\cos\theta}$,

$$(1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + (n^2-1)y = 0.$$

Solution by Professor MATHEWS, B.A.; Professor NASH, M.A.; *and others.*

If $x = \sin\theta$, and $y = \dfrac{\cos n\theta}{\cos\theta}$ or $\dfrac{\sin n\theta}{\cos\theta}$, we have

$$y(1-x^2)^{\frac{1}{2}} = \cos(n\sin^{-1}x) \quad \text{or} \quad \sin(n\sin^{-1}x),$$

$$(1-x^2)^{\frac{1}{2}}\frac{dy}{dx} - \frac{x}{(1-x^2)^{\frac{1}{2}}}y = -\frac{n}{(1-x^2)^{\frac{1}{2}}}\sin(n\sin^{-1}x)$$

or

$$\frac{n}{(1-x^2)^{\frac{1}{2}}}\cos(n\sin^{-1}x),$$

$$(1-x^2)\frac{dy}{dx} - xy = -n\sin(n\sin^{-1}x) \quad \text{or} \quad n\cos(n\sin^{-1}x);$$

$$(1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = -n^2\frac{\cos(n\sin^{-1}x)}{(1-x^2)^{\frac{1}{2}}},$$

or

$$-n^2\frac{\sin(n\sin^{-1}x)}{(1-x^2)^{\frac{1}{2}}} = -n^2y,$$

or

$$(1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + (n^2-1)y = 0.$$

Let $y = \Sigma a_r\dfrac{x^r}{r!}$, then, substituting in this equation, the coefficient of x^r will be

$$\frac{a_{r+2}}{r!} - \frac{a_r}{r-2!} - \frac{3a_r}{r-1!} + (n^2-1)\frac{a_r}{r!} = 0,$$

or

$$a_{r+2} = a_r(r^2-r+3r+1-n^2) = [(r+1)^2-n^2]a_r.$$

But, when $y = \dfrac{\cos n\theta}{\cos\theta}$, $a_0 = 1$, $a_1 = 0$, whence $a_3 = 0$, $a_5 = 0, \ldots$,

$$a_2 = 1^2-n^2,\ a_4 = (1^2-n^2)(3^2-n^2),\ \&c.,$$

$$\frac{\cos n\theta}{\cos\theta} = 1 + (1^2-n^2)\frac{\sin^2\theta}{2!} + (1^2-n^2)(3^2-n^2)\frac{\sin^4\theta}{4!} + \ldots$$

When $y = \dfrac{\sin n\theta}{\cos\theta}$, $a_0 = 0$, $a_1 = n$; whence $a_2 = 0$, $a_4 = 0, \ldots$,

$$a_3 = (2^2-n^2)a_1 = n(2^2-n^2),\ a_5 = n(2^2-n^2)(4^2-n^2),\ \&c.,$$

and

$$\frac{\sin n\theta}{\cos\theta} = n\sin\theta + n(2^2-n^2)\frac{\sin^3\theta}{3!} + n(2^2-n^2)(4^2-n^2)\frac{\sin^5\theta}{5!} + \ldots$$

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Since we have taken $\sin^{-1}(0)$ to be 0, we must always take $\sin^{-1}(x)$ or θ to be an angle between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$; but n may have any real value, both series being always convergent except at the limits $\pm\frac{1}{2}\pi$. Dividing both members of the last equation by n, and then diminishing n indefinitely, we get as the limiting form

$$\frac{\theta}{\cos\theta} = \sin\theta + \tfrac{2}{3}\sin^3\theta + \frac{2.4}{3.5}\sin^5\theta + \frac{2.4.6}{3.5.7}\sin^7\theta + \ldots,$$

which of course may be got by differentiating the known equation

$$\tfrac{1}{2}\theta^2 = \tfrac{1}{2}\sin^2\theta + \frac{2}{3}\frac{\sin^4\theta}{4} + \frac{2.4}{3.5}\frac{\sin^6\theta}{6} + \ldots$$

Multiplying by $\cos^2\theta$, we get

$$\theta\cos\theta = \sin\theta - \tfrac{2}{3}\sin^3\theta - \frac{2}{3.5}\sin^5\theta - \frac{2.4}{3.5.7}\sin^7\theta - \ldots$$

and integrating from 0 to $\frac{1}{2}\pi$,

$$\tfrac{1}{2}\pi - 1 = 1 - \frac{2}{3^2} - \frac{2^2.4}{3^2.5^2} - \frac{2^2.4^2.6}{3^2.5^2.7^2} - \ldots$$

or $$\tfrac{1}{2}\pi = 1 - \frac{1}{3^2} - 2\cdot\frac{2^2}{3^2.5^2} - 3\cdot\frac{2^2.4^2}{3^2.5^2.7^2} - 4\cdot\frac{2^2.4^2.6^2}{3^2.5^2.7^2.9^2} - \ldots$$

a very slowly converging series.

By multiplying each equation by $\cos\theta$ and then integrating, we get the well-known equations

$$\sin n\theta = n\sin\theta + n\left(1^2 - n^2\right)\frac{\sin^3\theta}{3} + n\left(1^2 - n^2\right)\left(3^2 - n^2\right)\frac{\sin^5\theta}{5!} + \ldots$$

$$\cos n\theta = 1 - n^2\frac{\sin^2\theta}{2} - n^2\left(2^2 - n^2\right)\frac{\sin^4\theta}{4} - \ldots,$$

θ being as before an angle between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, and n any real quantity.

[illegible] By [illegible], M.A. Through a point P tangents PQ and PR are drawn to a parabola, and across the chord QR the tangent at any point of the arc is drawn, meeting PQ and PR respectively in [illegible] and [illegible]. Prove that the portion of the tangent intercepted between PQ and PR subtends a right angle at the focus of the parabola.

Solution by [illegible], M.A. [illegible], M.A., and others.

The coordinates of the points of the curve have the form $x = at^2$, $y = 2at$, so that the chord passing through the points

[illegible]

Hence the direction of the common chord, which is at right angles to the line joining the vertex to

[illegible]

But this is also the direction of the tangent at the point [illegible]. The required [illegible] from the result.

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8260. (By Rev. T. R. Terry, M.A.)—If the normal at P to the parabola $y^2 = 4ax$ meet the curve again at Q, and if $PQ^2 : 4SP^2 = SP : SA$, then $PQ = 6a3^{\frac{1}{2}}$.

Solution by R. Knowles, B.A.; A. Gordon; *and others.*

Let (h, k), $\left(\frac{4a^2}{k^2}, -\frac{4a^2}{k}\right)$ be the coordinates of the pole of PQ and of P; then $PQ^2 = \frac{(k^2+4a^2)^3}{a^2k^2}$, $SP = a + \frac{4a^2}{k^2}$, and, by the question,

$$aPQ^2 = 4SP^2, \text{ therefore } k^2 = 2a^2 \text{ and } PQ = 6\sqrt{3}\,a.$$

8356. (By Professor G. B. Mathews, B.A.)—Prove that (1)

$$\frac{k+(1-ki)\,e^{\phi i}}{1+ki+ke^{\theta i}} = e^{\phi i}, \text{ where } \tan\phi = \frac{\sin\theta - 2k\cos\theta - 2k^2(1+\sin\theta)}{\cos\theta + 2k\,(1+\sin\theta)};$$

and (2), $$\tan\tfrac{1}{2}(\theta-\phi) = \frac{k\,(1+\sin\theta)}{1+k\cos\theta}.$$

Solution by W. J. Barton, M.A.; G. G. Storr, B.A.; *and others.*

(1) We have $$\frac{e^{2i\phi}-1}{e^{2i\phi}+1} = \frac{i\sin\theta - 2ki\cos\theta - 2k^2 i\,(1+\sin\theta)}{\cos\theta + 2k\,(1+\sin\theta)};$$

therefore
$$e^{2i\phi} = \frac{(\cos\theta + i\sin\theta)(1-2ki) + 2k - 2k^2 i - k^2\,(e^{i\theta} - e^{-i\theta})}{(\cos\theta - i\sin\theta)(1+2ki) + 2k + 2k^2 i + k^2\,(e^{i\theta} - e^{-i\theta})}$$
$$= \frac{e^{2i\theta}(1-ki)^2 + 2k\,(1-ki)\,e^{i\theta} + k^2}{(1+ik)^2 + 2k\,(1+ki)\,e^{i\theta} + k^2\,e^{2i\theta}};$$

therefore
$$e^{i\phi} = \frac{k+(1-ki)\,e^{i\theta}}{1+ki+ke^{i\theta}}.$$

(2) $$\tan\tfrac{1}{2}(\theta-\phi) = \frac{e^{\frac{1}{2}i(\theta-\phi)}-1}{i\left[e^{\frac{1}{2}i(\theta-\phi)}+1\right]} = \frac{e^{\theta} - e^{i\phi}}{i\,(e^{i\theta}+e^{i\phi})}$$
$$= \left\{e^{i\theta} - \frac{k+(1-ki)\,e^{i\theta}}{1+ki+ke^{i\theta}}\right\} \Big/ i\left\{e^{i\theta} + \frac{k+(1-ki)\,e^{i\theta}}{1+ki+ke^{i\theta}}\right\}$$
$$= k\left(\frac{e^{i\theta}+e^{-i\theta}}{2i}+1\right) \Big/ \left(k\frac{e^{i\theta}+e^{-i\theta}}{2}+1\right) = \frac{k\,(1+\sin\theta)}{1+k\cos\theta}.$$

8376. (By F. Morley, B.A.)—There are two points whose distances from fixed points A, B, C are in the ratios $p : q : r$; show that the line joining them goes through a fixed point, whatever be the values of the ratios.

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Since we have taken $\sin^{-1}(0)$ to be 0, we must always take $\sin^{-1}(x)$ or θ to be an angle between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$; but n may have any real value, both series being always convergent except at the limits $\pm\frac{1}{2}\pi$. Dividing both members of the last equation by n, and then diminishing n indefinitely, we get as the limiting form

$$\frac{\theta}{\cos\theta} = \sin\theta + \tfrac{2}{3}\sin^3\theta + \frac{2\cdot4}{3\cdot5}\sin^5\theta + \frac{2\cdot4\cdot6}{3\cdot5\cdot7}\sin^7\theta + \dots,$$

which of course may be got by differentiating the known equation

$$\tfrac{1}{2}\theta^2 = \tfrac{1}{2}\sin^2\theta + \frac{2}{3}\frac{\sin^4\theta}{4} + \frac{2\cdot4}{3\cdot5}\frac{\sin^6\theta}{6} + \dots$$

Multiplying by $\cos^2\theta$, we get

$$\theta\cos\theta = \sin\theta - \tfrac{1}{3}\sin^3\theta - \frac{2}{3\cdot5}\sin^5\theta - \frac{2\cdot4}{3\cdot5\cdot7}\sin^7\theta - \dots$$

and integrating from 0 to $\frac{1}{2}\pi$,

$$\tfrac{1}{4}\pi - 1 = 1 - \frac{2}{3^2} - \frac{2^2\cdot4}{3^2\cdot5^2} - \frac{2^2\cdot4^2\cdot6}{3^2\cdot5^2\cdot7^2} - \dots$$

or $$\tfrac{1}{4}\pi = 1 - \frac{1}{3^2} - 2\cdot\frac{2^2}{3^2\cdot5^2} - 3\cdot\frac{2^2\cdot4^2}{3^2\cdot5^2\cdot7^2} - 4\cdot\frac{2^2\cdot4^2\cdot6^2}{3^2\cdot5^2\cdot7^2\cdot9^2} - \dots$$

a very slowly converging series.

Multiplying each equation by $\cos\theta$ and then integrating, we get the well-known equations

$$\sin n\theta = n\sin\theta + n(1^2-n^2)\frac{\sin^3\theta}{3!} + n(1^2-n^2)(3^2-n^2)\frac{\sin^5\theta}{5!} + \dots$$

$$\cos n\theta = 1 - n^2\frac{\sin^2\theta}{2!} - n^2(2^2-n^2)\frac{\sin^4\theta}{4!} - \dots,$$

θ being as before an angle between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, and n any real quantity.

8229. (By J. Brill, M.A.)—Through a point P tangents PQ and PR are drawn to a parabola, and circles are drawn through the focus touching the parabola at Q and R respectively; prove that the common chord of these circles passes through the centroid of the triangle PQR.

Solution by R. J. Brevor, B.A.; Rev. T. Galliers, M.A.; *and others.*

The coordinates of the centre of the circle touching $y = m_1x + a/m_1$ at P, and passing through the focus, are

$$\frac{a}{2}\left(1 + \frac{3}{m_1^2}\right), \quad \frac{a}{2}\left(\frac{3}{m_1} - \frac{1}{m_1^3}\right).$$

Hence the direction of the common chord, which is at right angles to the line joining the centres, is

$$\tan^{-1}\frac{(3/m_1) + (3/m_2)}{-3 + (1/m_1^2) + (1/m_1m_2) + (1/m_2^2)}.$$

But this is also the direction of the line joining S $(a, 0)$ to the centroid $\left\{\frac{a}{3}\left(\frac{1}{m_1^2} + \frac{1}{m_1m_2} + \frac{1}{m_2^2}\right), a\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\right\}$. Hence the result.

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8260. (By Rev. T. R. Terry, M.A.)—If the normal at P to the parabola $y^2=4ax$ meet the curve again at Q, and if $PQ^2 : 4SP^2 = SP : SA$, then $PQ = 6a3^{\frac{1}{2}}$.

Solution by R. Knowles, B.A.; A. Gordon; *and others.*

Let (h, k), $\left(\frac{4a^2}{k^2}, -\frac{4a^2}{k}\right)$ be the coordinates of the pole of PQ and of P; then $PQ^2 = \frac{(k^2+4a^2)^3}{a^2k^2}$, $SP = a + \frac{4a^2}{k^2}$, and, by the question,

$$aPQ^2 = 4SP^2, \text{ therefore } k^2 = 2a^2 \text{ and } PQ = 6\sqrt{3}\,a.$$

8356. (By Professor G. B. Mathews, B.A.)—Prove that (1)

$$\frac{k+(1-ki)\,e^{\phi i}}{1+ki+ke^{\phi i}} = e^{\psi i}, \text{ where } \tan\phi = \frac{\sin\theta - 2k\cos\theta - 2k^2(1+\sin\theta)}{\cos\theta + 2k\,(1+\sin\theta)};$$

and (2), $$\tan\tfrac{1}{2}(\theta-\phi) = \frac{k\,(1+\sin\theta)}{1+k\cos\theta}.$$

Solution by W. J. Barton, M.A.; G. G. Storr, B.A.; *and others.*

(1) We have $$\frac{e^{2i\phi}-1}{e^{2i\phi}+1} = \frac{i\sin\theta - 2ki\cos\theta - 2k^2 i\,(1+\sin\theta)}{\cos\theta + 2k\,(1+\sin\theta)};$$

therefore
$$e^{2i\phi} = \frac{(\cos\theta + i\sin\theta)(1-2ki) + 2k - 2k^2 i - k^2\,(e^{i\theta} - e^{-i\theta})}{(\cos\theta - i\sin\theta)(1+2ki) + 2k + 2k^2 i + k^2\,(e^{i\theta} - e^{-i\theta})}$$
$$= \frac{e^{2i\theta}\,(1-ki)^2 + 2k\,(1-ki)\,e^{i\theta} + k^2}{(1+ik)^2 + 2k\,(1+ki)\,e^{i\theta} + k^2 e^{2i\theta}};$$

therefore
$$e^{i\psi} = \frac{k+(1-ki)\,e^{i\theta}}{1+ki+ke^{i\theta}}.$$

(2) $$\tan\tfrac{1}{2}(\theta-\phi) = \frac{e^{i(\theta-\phi)}-1}{i\,[e^{i(\theta-\phi)}+1]} = \frac{e^{\theta} - e^{i\phi}}{i\,(e^{i\theta}+e^{i\phi})}$$
$$= \left\{e^{i\theta} - \frac{k+(1-ki)\,e^{i\theta}}{1+ki+ke^{i\theta}}\right\} \Big/ i\left\{e^{i\theta} + \frac{k+(1-ki)\,e^{i\theta}}{1+ki+ke^{i\theta}}\right\}$$
$$= k\left(\frac{e^{i\theta}+e^{-i\theta}}{2i}+1\right) \Big/ \left(k\,\frac{e^{i\theta}+e^{-i\theta}}{2}+1\right) = \frac{k\,(1+\sin\theta)}{1+k\cos\theta}.$$

8376. (By F. Morley, B.A.)—There are two points whose distances from fixed points A, B, C are in the ratios $p : q : r$; show that the line joining them goes through a fixed point, whatever be the values of the ratios.

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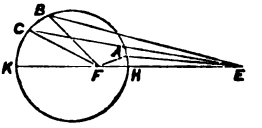
Solution by Dr. CURTIS; F. R. J. HERVEY, M.A.; *and others.*

Let E and F be the two points, then

$$EA : EB : EC = p : q : r = FA : FB : FC,$$

therefore $$\frac{EA}{EB} = \frac{FA}{FB}, \quad \frac{EB}{EC} = \frac{FB}{FC};$$

therefore $$\frac{EA}{FA} = \frac{EB}{FB} = \frac{EC}{FC};$$

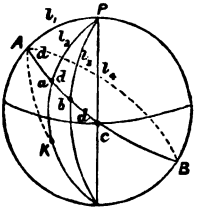


hence, if EF be cut in H, and produced to K, in this ratio, whatever it may be, the circle on HK as diameter will pass through the three fixed points A, B, C, and is therefore a fixed circle; and EF, being a diameter of this circle, will always pass through its centre, a fixed point.

8289. (By C. MORGAN, B.A., R.N.)—If a great circle on the earth be divided in any number of equal parts, and l_1, l_2, l_3 ... be the colatitudes of the points of section, prove that $\Sigma \cos^{-1}\left(\frac{\cos l_n - \cos l_{n-2} \cos 2l_{n-1}}{\sin l_{n-2} \sin 2l_{n-1}}\right) = 2\pi$.

Solution by W. J. C. SHARP, M.A.; *the* PROPOSER; *and others.*

Let the equal parts be each $= d$; and let P be the pole of the earth, AB the great circle, $PA = l_1$, $Pa = l_2$, $APd = \lambda_1$, $aPb = \lambda_2$; then



$$\cos PaA = \frac{\cos l_1 - \cos l_2 \cos d}{\sin l_2 \sin d},$$

$$\cos Pab = \frac{\cos l_2 - \cos l_2 \cos d}{\sin l_2 \sin d},$$

therefore $$\cos l_1 + \cos l_3 = 2 \cos l_2 \cos d.$$

$$\cos \lambda_1 = \frac{\cos d - \cos l_1 \cos l_2}{\sin l_1 \sin l_2}$$

$$= \frac{\cos l_1 + \cos l_3 - 2 \cos l_1 \cos^2 l_2}{2 \sin l_1 \sin l_2 \cos l_2} = \frac{\cos l_3 - \cos l_1 \cos 2l_2}{\sin l_1 \sin 2l_2} \quad \ldots\ldots(a),$$

$$\lambda_1 = \cos^{-1}\left(\frac{\cos l_3 - \cos l_1 \cos 2l_2}{\sin l_1 \sin 2l_2}\right),$$

therefore $$\Sigma \cos^{-1}\left(\frac{\cos l_n - \cos l_{n-2} \cos 2l_{n-1}}{\sin l_{n-2} \sin 2l_{n-1}}\right) = \lambda_1 + \lambda_2 + \ldots = 2\pi.$$

[Equation (a) can be proved directly by taking $aK = l_2$ and joining AK by arc of great circle; then $AK = l_3$, $\therefore \cos \lambda_1 = \frac{\cos l_3 - \cos l_1 \cos 2l_2}{\sin l_1 \sin 2l_2}$].

2231. (By Professor SYLVESTER, F.R.S.)—If a system of conics touch a given straight line at a given point at which they have all the same curvature, and pass through a second given point, show that, on taking as

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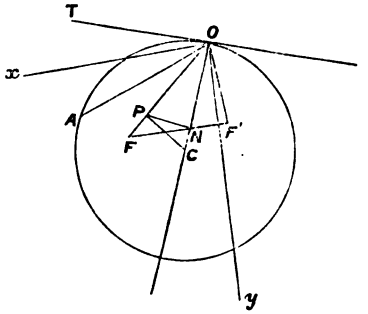
axis of coordinates the bisectors of the angle between the given line and the line joining the two given points, the locus of the foci of the conics so taken consists of the two circular cubics

$$2by\,(x^2+y^2)-(ax+by)^2=0,\quad 2ax\,(x^2+y^2)-(ax+by)^2=0,$$

where a, b are the x, y coordinates of the second given point.

Solution by Professor NEUBERG.



Soient F, F′ les foyers d'une conique osculatrice au cercle fixe C au point donné O et coupant ce cercle au point donné A. On sait (1) que la corde OA et la tangente OT sont également inclinées sur l'axe FF′; (2) que le centre C du cercle de courbure s'obtient en menant par le pied N de la normale une perpendiculaire NP à la normale et par le point de rencontre P de cette droite avec le rayon vecteur OF une perpendiculaire PC à ce rayon.

Soient Ox, Oy les bissectrices de l'angle TOA, $xOA=\alpha$, $xOF=\omega$, $OF=\rho$, $OC=R$. Nous aurons $ON=R\sin^2(\alpha+\omega)$. Les distances de N et F à Ox étant égales entre elles, l'équation du lieu de F en coordonnées polaires est

$$\rho\sin\omega=h\sin^2(\alpha+\omega)\cos\alpha;$$

d'ou l'on déduit l'équation en coordonnées rectangulaires

$$y\,(x^2+y^2)=R\,(x\sin\alpha+y\cos\alpha)^2\cos\alpha,$$

ou, à cause de $\sin\alpha=b/(a^2+b^2)^{\frac{1}{2}}$, $\cos\alpha=a/(a^2+b^2)^{\frac{1}{2}}$, $2R=OA/\sin 2\alpha$,

$$4by\,(x^2+y^2)=(ax+by)^2.$$

L'axe focal FF′ peut être parallèle à Oy, ce qui donne une seconde cubique.

8636. (By Professor MAHENDRA NATH RAY, M.A., LL.B.)—Show that the following equations are satisfied by the *same* value of x, and find this value:—

$$ax\,(x^2-a^2)^{\frac{1}{2}}+bx\,(x^2-b^2)^{\frac{1}{2}}+cx\,(x^2-c^2)^{\frac{1}{2}}=2abc,$$

$$2\,(x^2-a^2)^{\frac{1}{2}}(x^2-b^2)^{\frac{1}{2}}(x^2-c^2)^{\frac{1}{2}}=x\,(a^2+b^2+c^2-2x^2).$$

Solutions by (1) Professor CAYLEY, F.R.S.; (2) Professor NEUBERG.

1. The second equation rationalised gives

$$4x^6-4x^4(a^2+b^2+c^2)+4x^2(b^2c^2+c^2a^2+a^2b^2)-4a^2b^2c^2$$
$$=4x^6-4x^4(a^2+b^2+c^2)+x^2(a^2+b^2+c^2)^2;$$

that is,

$$\nabla x^2=4a^2b^2c^2.$$

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If, for shortness, $\nabla = -a^4 - b^4 - c^4 + 2b^2c^2 + 2c^2a^2 + 2a^2b^2$,

we thence find $\nabla(x^2 - a^2) = a^2(-a^2 + b^2 + c^2)^2$,

$$\nabla(x^2 - b^2) = b^2(a^2 - b^2 + c^2)^2, \quad \nabla(x^2 - c^2) = c^2(a^2 + b^2 - c^2)^2,$$

and therefore also $\nabla^2 a^2 x^2 (x^2 - a^2) = 4a^6 b^2 c^2 (-a^2 + b^2 + c^2)^2$, &c.

Or, assuming the sign of the square roots,

$$\nabla ax(x^2 - a^2)^{\frac{1}{2}} = 2abc(-a^4 + a^2b^2 + a^2c^2), \quad \nabla bx(x^2 - b^2)^{\frac{1}{2}} = 2abc(+a^2b^2 - b^4 + b^2c^2),$$

$$\nabla cx(x^2 - c^2)^{\frac{1}{2}} = 2abc(a^2c^2 + b^2c^2 - c^4),$$

whence, adding, the whole divides by ∇ and we have

$$ax(x^2 - a^2)^{\frac{1}{2}} + bx(x^2 - b^2)^{\frac{1}{2}} + cx(x^2 - c^2)^{\frac{1}{2}} = 2abc,$$

the second equation. Observe that the second equation rationalised gives an equation of the form $(x^2, 1)^4 = 0$; the foregoing value $x^2 = 4a^2b^2c^2 / \Delta$ is thus one of the four values of x^2.

2. *Otherwise*:—On trouve les deux équations en cherchant le rayon x du cercle circonscrit à un triangle ABC dont les côtés sont $2a$, $2b$, $2c$. En effet, si A′, B′, C′ sont les milieux des côtés de ABC et O le centre du cercle, on a

$$\text{tg A} = \text{tg A'OB} = \frac{\text{OA'}}{\text{A'B}} = \frac{(x^2 - a^2)^{\frac{1}{2}}}{a}, \quad \cos \text{A} = \frac{\text{OA'}}{\text{OB}} = \frac{(x^2 - a^2)^{\frac{1}{2}}}{x}, \ \ldots\ldots;$$

substituons ces valeurs dans les identités

$$\text{tg A} + \text{tg B} + \text{tg C} = \text{tg A tg B tg C},$$

$$1 - \cos^2 \text{A} - \cos^2 \text{B} - \cos^2 \text{C} + 2 \cos \text{A} \cos \text{B} \cos \text{C} = 0 \ \ldots\ldots$$

Pour avoir x, on peut observer que

$$x = \frac{abc}{4S} = \frac{abc}{4\{p(p-a)(p-b)(p-c)\}^{\frac{1}{2}}}, \text{ où } 2p = a + b + c.$$

8510. (By Professor Hermite, LL.D.)—Trouver l'expression la plus générale d'un polynôme F (x), de degré $2m$, tel qu'on ait :

$$x^m \text{F}\left(\frac{1}{x}\right) = \text{F}(x), \quad (x+1)^{2m} \text{F}\left(\frac{1-x}{1+x}\right) = 2^m \text{F}(x) \quad \ldots\ldots(1, 2).$$

Solution by the Proposer.

1. F (x) ne contient que des puissances paires de x. En effet, changer dans (2) x en $\frac{1}{x}$, on a après avoir chassé le dénominateur :

$$(x+1)^{2m} \text{F}\left(-\frac{1-x}{1+x}\right) = 2^m x^{2m} \text{F}\left(\frac{1}{x}\right) \text{ donc } = 2^m \text{F}(x)$$

$$= (x+1)^{2m} \text{F}\left(\frac{1-x}{1+x}\right);$$

et, si vous faites : $\frac{1-x}{1+x} - z$, vous avez, en effet, F $(-z)$ = F (z).

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2. Je dis que, si $F(x)$ s'annule avec x, il contient le facteur $x^2(1-x^2)^2$. En effet, soit dans (2) $x=0$, vous en concluez : $F(1)=0$; par conséquent $F(x)$, qui ne contient que des puissances paires, admet le facteur $x^2(1-x^2)$. Soit donc $F(x)=x^2(1-x^2)G(x)$, (1) vous donnera :

$$x^{2m-6}G\left(\frac{1}{x}\right)=-G(x),$$

par conséquent $G(x)$ s'évanouit pour $x=1$, et $x=-1$.

3. Soit : $F(x)=x^2(1-x^2)^2H(x)$ le nouveau polynôme $H(x)$, en vertu de (1) et (2), remplit les conditions :

$$x^{2m-8}H\left(\frac{1}{x}\right)=H(x),\quad (x+1)^{2m-8}H\left(\frac{1-x}{1+x}\right)=2^{m-}H(x)\ ;$$

par conséquent $H(x)$ reproduit les équations caractéristiques (1, 2) en y changeant m en $m-4$.

4. Soit : $F_1(x)=F(x)-A(x^2+1)^m$. A étant un coefficient constant arbitraire, on vérifie facilement qu'on a :

$$x^{2m}F_1\left(\frac{1}{x}\right)=F_1(x),\quad (x+1)^{2m}F_1\left(\frac{1-x}{1+x}\right)=2^mF_1(x).$$

Disposant donc de A, de manière que $F_1(x)$ s'annule pour $x=0$, on le ramène ainsi au produit d'un polynôme de même nature de degré $2(m-4)$, multiplié par le facteur $x^2(1-x^2)^2$. Donc de proche en proche vous parvenez à l'expression cherchée à savoir :

$$F(x)=A(x^2+1)^{2m}+x^2(1-x^2)^2B(x^2+1)^{m-4}+x^4(1-x^2)^4C(x^2+1)^{m-8}+\ldots$$
$$\ldots+x^{2i}(1-x^2)^{2i}L(x^2+1)^{m-4i}.$$

Si l'on demande les polynômes qui satisfont aux conditions :

$$x^{2m}F\left(\frac{1}{x}\right)=F(x),\quad (x+1)^{2m}F\left(\frac{1-x}{1+x}\right)=-2^mF(x),$$

on remarque que ces polynômes s'annulent pour les valeurs qui donnent : $\frac{1-x}{1+x}=x$, c'est-à-dire $x^2+2x-1=0$. Et comme dans ce second cas comme dans le précédent $F(-x)=F(x)$, vous avez aussi le facteur x^2-2x-1 ; on peut donc poser : $F(x)=(x^2+2x-1)(x^2-2x-1)G(x)$; cela étant, on obtient sans peine :

$$x^{2m-4}G\left(\frac{1}{x}\right)=G(x),\quad (1+x)^{2m-4}G\left(\frac{1-x}{1+x}\right)=2^{m-2}G(x),$$

ce sont les conditions (1) et (2), en y changeant m en $m-2$.

8217. (By Professor CATALAN.)—On donne une suite indéfinie de termes, dont les deux premiers sont $A_0=1$, $A_1=4$, et dont les autres, à partir de $n=2$, sont donnés par la loi de récurrence,

$$n^2A_n-4(3n^2-3n+1)A_{n-1}+32(n-1)^2A_{n-2}=0\ ;$$

prouver que tous ces termes sont des *nombres entiers*.

Solution by W. J. C. SHARP, M.A.

It is known (BOOLE's *Differential Equations*, p. 431, ed. 1) that if

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$u = \Sigma . u_m x^m$, where the law of formation of the successive coefficients is

$$f_0(m)\, u_m + f_1(m)\, u_{m-1} + \ldots + f_n(m)\, u_{m-n} = 0,$$

the value of u is the solution of the equation

$$f_0(\mathrm{D})\, u + f_1(\mathrm{D})\, e^{\theta} u + \ldots + f_n(\mathrm{D})\, e^{n\theta} u = 0.$$

Hence the series $A_0 + A_1 x + A_2 x^2 + \&c.$ is the solution of the equation

$$\mathrm{D}^2 u - 4(3\mathrm{D}^2 - 3\mathrm{D} + 1)\, e^{\theta} u + 32(\mathrm{D}-1)^2 e^{2\theta} u = 0$$

(the constants being evaluated so that $A_0 = 1$ and $A_1 = 4$). This equation is equivalent to $\mathrm{D}^2 u - 4e^{\theta}(3\mathrm{D}^2 + 3\mathrm{D} + 1)\, u + 32e^{2\theta}(\mathrm{D}+1)^2 u = 0$,

or $$\{\mathrm{D}(\mathrm{D}-1) + \mathrm{D}\}\, u - 4e^{\theta}\{3\mathrm{D}(\mathrm{D}-1) + 6\mathrm{D} + 1\}\, u + 32e^{2\theta}\{\mathrm{D}(\mathrm{D}-1) + 3\mathrm{D} + 1\}\, u = 0,$$

or $$\left(x^2\frac{d^2}{dx^2} + x\frac{d}{dx}\right)u - 4x\left\{3x^2\frac{d^2}{dx^2} + 6x\frac{d}{dx} + 1\right\}u + 32x^2\left\{x^2\frac{d^2}{dx^2} + 3x\frac{d}{dx} + 1\right\}u = 0,$$

i.e., $$(x - 12x^2 + 32x^3)\frac{d^2u}{dx^2} + (1 - 24x + 96x^2)\frac{du}{dx} - 4(1-8x)\, u = 0,$$

that is, $$\frac{d}{dx}\left\{x(1-4x)(1-8x)\frac{du}{dx}\right\} = 4(1-8x)\, u;$$

let $y = x - 4x^2$, therefore $\frac{dy}{dx} = 1 - 8x$,

and the equation becomes $$\frac{d}{dy}\left\{y(1-16y)\frac{du}{dy}\right\} = 4u.$$

Now assume $u = a_0 + a_1 y + a_2 y^2 + \&c.$ (and this involves $a_0 = 1$ and $a_1 = 0$, since $A_0 = 1$ and $A_1 = 4$), therefore

$$y(1-16y)\frac{du}{dy} = a_1 y + 2a_2 y^2 + \ldots + n a_n y^n + \&c. - 16a_1 y^2 + \ldots + 16(n-1)\, a_{n-1} y^n + \&c.,$$

and therefore $$\frac{d}{dy}\left\{y(1-16y)\frac{du}{dy}\right\} = a_1 + 2^2 a_2 y + \ldots + n^2 a_n y^{n-1} + \&c. - 16 \cdot 2a_2 y + \ldots + 16n(n-1)\, a_{n-1} y^{n-1} + \&c.,$$

which is identical with $4(a_0 + a_1 y + \ldots + a_{n-1} y^{n-1} + \&c.)$; $\therefore\ a_1 = 4a_0$, $2^2 a_2 - 16 \cdot 2a_1 = 4a_1$, &c., &c., &c.; $n^2 a_n - 16n(n-1)\, a_{n-1} = 4a_{n-1}$, or $n^2 a_n = 4(2n-1)^2 a_{n-1}$, and therefore, since $a_0 = 1$, $a_1 = 4$ as it should, and $a_n = 2^{2n}\,\frac{1^2 \cdot 3^2 \ldots (2n-1)^2}{1^2 \cdot 2^2 \ldots n^2}$, a whole number; and therefore, since the series $A_0 + A_1 x + A_2 x^2 + \&c.$ is identical with $1 + 4(x - 4x^2) + a_2(x - 4x^2)^2 + \&c.$, A_0, A_1, &c. are whole numbers.

8185. (By Professor Neuberg.)—On donne la base BC d'un triangle ABC, et le sommet A se meut sur une droite donnée BY. Trouver (1) les lieux du centre de gravité et du cercle des neuf points (droites); (2) l'enveloppe de la droite qui joint ces points (conique); (3) les lieux des

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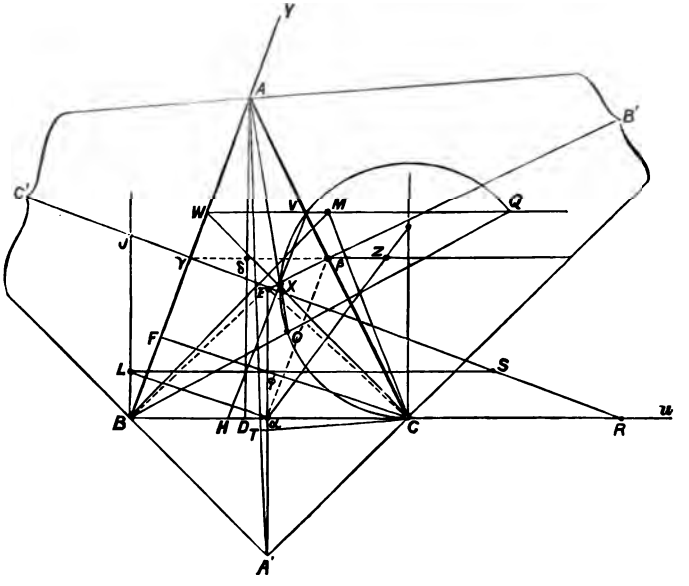
sommets B′, C′ du triangle A′B′C′, que forment les tangentes en A, B, C au cercle ABC (parallèle à BC et conique) ; (4) l'enveloppe de B′C′ (parabole de foyer C) ; (5) l'enveloppe de la symédiane AA′ (conique de foyer C ; (6) le lieu décrit par le centre K des symédianes (conique) ; (7) les lieux décrits par les points de Brocard (cercle et polaire de conique).

I. *Solution by* CHARLOTTE A. SCOTT, D.Sc.

1. To find locus of centroid (G).—Let α be mid-point of BC, therefore $\alpha G = \frac{1}{3}\alpha A$, therefore locus of G is a straight line, parallel to BY.



To find locus of mid-centre (N).—Let CF be perpendicular to BY. Then F and α are points on mid-circle, and they are fixed. Therefore locus of mid-centre is a straight line, perpendicular to and bisecting αF.

2. To find envelope of NG.—Let I be circumcentre, P the orthocentre. Locus of I is straight line through α, perpendicular to BC ; locus of P is line CF ; locus of G, as of N, is also a straight line. The range IGNP is harmonic, *i.e.*, the variable line GN meets four fixed lines so as to be cut harmonically ; it therefore envelopes a conic.

3. To find locus of B′.—Draw B′U perpendicular to BC. Then

$$\angle B'CU = \angle A = \angle FAC, \text{ therefore } \angle CB'U = \angle ACF,$$

therefore $$B'U : CF = B'C : CA.$$

But $\angle B'CA = \angle B$, therefore B′C : CA is constant, therefore L′U is constant, *i.e.*, locus of B′ is a straight line, parallel to BC.

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To find locus of C′.—Draw BJ perpendicular to BC, αL perpendicular to BA, LS parallel to BC. Then C′BI, JBR are right angles, and Bγ is perpendicular to C′R, therefore C′γ . γI = Jγ . γR, *i.e.* Jγ : C′γ = γI : γR, therefore C′J : C′γ = RI : γR. But Lα is parallel to JR, therefore JS = IR, therefore C′J : JS = C′γ : γR. We have then

$$\{\infty\, C'JS\} = \{\infty\, C'\gamma R\},$$

i.e., the two pencils (LS, LJ, LC′, and parallel to RC′), (BC, BA, BC′, and parallel to RC′) are homographic. If therefore a hyperbola be described passing through L, touching BA at B, and having its asymptotes parallel to RI, BC, this will be the locus of C′.

4. To find envelope of B′C′.—The $\angle$ CB′C′ = supplement of AIC = supplement 2B = constant. Therefore, as B′ moves along the fixed line which is its locus, B′C′ envelopes a parabola, focus C.

5. To find envelope of AA′.—Draw CT perpendicular to AA′. Then, because AFTC is a circle, therefore $\angle$ FTA = $\angle$ FCA = complement A; and because CαTA′ is a circle, therefore $\angle$ αTA = $\angle$ αCA′ = $\angle$ A, therefore $\angle$ FTα = $\angle$ FTA + $\angle$ ATα = complement A + A = a right angle, therefore locus of T is circle on Fα as diameter, *i.e.*, AA′ envelopes a conic, focus C.

6. To find locus of Symmedian point (K).—ϕ, δ being bisections of CF, AD, K is intersection of $\alpha\delta$, $\gamma\phi$. Draw αZ parallel to Bδ, meeting $\beta\gamma$ in Z, therefore δZ = Bα = $\gamma\beta$, therefore $\gamma\delta$: $\delta\beta$ = Zβ : $\beta\delta$, therefore

$$\{\gamma\delta\beta\,\infty\} = \{Z\beta\delta\,\infty\},$$

i.e., the two pencils (BA, Bδ. Bβ, BC) and (αZ, $\alpha\beta$, $\alpha\delta$, αC) are homographic. Now $\angle$ ABδ = $\angle$ CBϕ and $\angle$ ABβ = $\angle$ CBK, therefore the pencils (BA, Bδ, Bβ, BC) and (BC, Bϕ, BK, BA) are homographic, therefore $\{B\,.\,C\phi KA\} = \{\alpha\,.\,Z\beta\delta C\}$, therefore K lies on a conic, passing through ϕ, touching αZ at α, BA at B.

7. To find locus of Brocard points (O and O′).—If O′ be second Brocard-point, then $\angle$ O′BA = $\angle$ O′CB, therefore circle through B, O′, C touches AB at B, *i.e.*, locus of O′ is a circle through B, C, touching AB at B.

If O be first Brocard-point: through O, C describe a circle touching BC at C, cutting CA, BO, AO in V, Q, X. Then $\angle$ QVC = $\angle$ QOC = supplement BOC = $\angle$ C, therefore QV is parallel to BC. Produce QV to meet AB in W, and a line through C (making $\angle$ BCM = B) in M. Then

$$\angle AXV = \angle VCO = \angle OAB,$$

therefore VX is parallel to AB; produce it to meet BC in H. In the circle we have a hexagon CVQOXC; CC (*i.e.*, tangent at C) and OQ meet in B, CV and OX meet in A. Therefore CX and QV must meet on line AB, *i.e.*, CX passes through W. Now WV = BH, CV = HM, and $\angle$ WVC = $\angle$ BHM, therefore BM = CW, and triangles BMC, CWB are equal. Now $\angle$ WCB = $\angle$ CVX = $\angle$ A, therefore the triangle BCM is similar to ABC, as also the triangle CMQ, therefore

$$QM : MC = MC : CB, \quad i.e., \quad MC^2 = QM\,.\,BC,$$

therefore locus of Q is a parabola, having its axis parallel to BC, and touching CM at C, *i.e.*, the locus of O is the inverse of a parabola, for

$$BO\,.\,BQ = BC^2.$$

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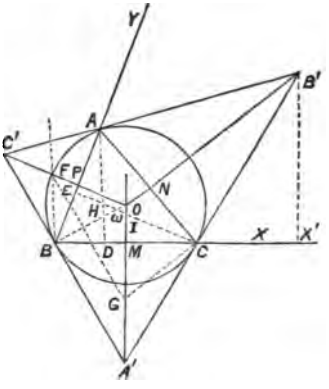
II. *Solution by the* PROPOSER.

1. Le centre de gravité décrit une parallèle à AB. Deux points du cercle des neuf points sont fixes, savoir le milieu M de BC et le pied de la hauteur CE abaissée de C...



2. Soient M, N, P les milieux des côtés, O le centre du cercle ABC, AD et CE deux hauteurs se coupant en H. Les droites AD, PO, lorsque A se déplace sur BY, engendrent des faisceaux parallèles homographiques; donc les points O et H déterminent sur les lignes fixes A'M, CE deux ponctuelles homographiques. Donc la droite HO a pour enveloppe une conique touchant CE et A'M.

3. Menons B'X' perpendiculaire à BC. On a

$$B'X' = B'C \sin A = \frac{NC \sin A}{\cos B} = \frac{b \sin A}{\cos B} = \frac{a \sin B}{\cos B};$$

donc B' décrit une parallèle à BC.

4. La droite AB' est vue de C sous l'angle B; A, B' décrivent des droites dont l'angle $= \pi - B$; donc l'enveloppe de AB' est une parabole ayant pour foyer C.

5. AA' est vue de C sous l'angle $ACA' = \pi - B$, et les points A, A' se meuvent sur deux droites; l'enveloppe est donc une conique de foyer C.

6. Soit K le centre des symédianes. La droite KC' est divisée harmoniquement par C et AB; donc K et C' décrivent deux figures homologiques.

7. L'un des points de Brocard a pour lieu la circonférence qui passe par B, C et touche AB.

Menons BF perpendiculaire à BC et rencontrant OP en F; menons aussi CG perpendiculaire à AC et rencontrant OM en G. Le point de Brocard ω est (1) sur la circonférence ayant pour centre F et passant par B et A; (2) sur la circonférence ayant pour centre G et passant par B et C. Ce point est donc symétrique de B par rapport à la droite FG. Mais lorsque A se meut sur BY, les droites OP, CA, CG (perpendiculaire à CA) engendrent des faisceaux homographiques; par suite, les points F et G marquent sur BF et OM des divisions homographiques. La droite FG enveloppe une conique touchant BF en B et OM en I (sur la hauteur CE). Par suite, ω décrit une courbe homothétique à la podaire de B par rapport à la conique.

8282. (By the EDITOR.)—If a coin be dropped at random over a grating composed of parallel equidistant wires in a horizontal plane, show by a *general* solution, that, according as the distance between the wire is equal to (1) the radius, (2) the diameter, (3) the circumference of the

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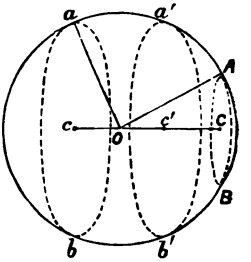
coin, the respective probabilities that the coin will not go through without striking are $p_1 = \frac{1}{8}\pi + \frac{1}{4}\sqrt{3}$, $p_2 = \frac{1}{4}\pi$, $p_3 = \frac{1}{4}$.

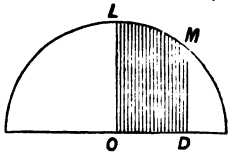
Solution by Rev. T. C. Simmons, M.A.

We must assume that the coin, during the interval under consideration, moves parallel to itself, and that every direction in space of its normal is equally probable, for on no other assumption will a reasonable solution be possible. This then being presumed, and the thickness of both coin and wires being supposed infinitesimal, the question may be completely solved, *without Integral Calculus*, as follows:—



Consider the coin as always inscribed in a sphere whose centre and radius are the same as its own. If, throughout the descent, the sphere remains clear of the wires, the coin will so remain likewise; if, on the other hand, the sphere strikes a wire, the clearing of the coin will depend upon circumstances, and will only occur when its plane is clear of the circle of intersection of the sphere by the vertical plane passing through the wire in question. Let A, B be the respectively highest and lowest points of this circle, and C its centre. Draw, in the vertical plane COA, Oa and Ob at right angles respectively to OA and OB, and let c be the centre of the circle of intersection of the sphere by the vertical plane ab perpendicular to OC. Then, if the corresponding parallel circle $a'b'$ be drawn on the other side of O, it will be evident that, when the plane of the coin intersects the circle AB, its central normal will intersect the spherical zone intercepted between the circles ab and $a'b'$. And, as the chance of the normal intersecting any portion of the spherical surface is proportional to the area thereof, the chance that the plane of the coin intersects AB will consequently be measured by the ratio of the aforesaid zonal area to that of the whole spherical surface, *i.e.*, by the ratio of Oc to the radius. Whence, taking r to denote this radius, and x to denote OC, the distance of the centre from the wire under consideration, the chance that the coin will strike the wire will be denoted by the very simple expression $(r^2 - x^2)^{\frac{1}{2}} / r$.

We have now two separate cases to consider.



(i.) Let the distance ($2d$) between two consecutive wires be less than the diameter of the coin. We have then to find the average value of $(r^2 - x^2)^{\frac{1}{2}} / r$ as x ranges from 0 to d. This is clearly the same as finding the average ratio, to the radius, of ordinates of a semicircle situated between LO and MD in the figure, where OL = r, and OD = d; and is represented by the ratio of the area LODM to the rectangle OD . OL, which is equal to $\frac{r}{2d}\sin^{-1}\frac{d}{r} + \frac{(r^2 - d^2)^{\frac{1}{2}}}{2r}$.

This, then, is the *general* expression for the required chance, which is

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easily evaluated in the following particular cases:—

(1) $d = r$, chance $= \frac{1}{4}\pi = \cdot 7854$.

(2) $d = (\frac{1}{2}\sqrt{3})\, r$, chance $= (\frac{1}{6}\sqrt{3})\, \pi + \frac{1}{4} = \cdot 8546$.

(3) $d = (\frac{1}{2}\sqrt{2})\, r$, chance $= (\frac{1}{8}\sqrt{2})\, \pi + \dfrac{1}{2\sqrt{2}} = \cdot 9089$.

(4) $d = \frac{1}{2}r$, chance $= \frac{1}{6}\pi + \frac{1}{4}\sqrt{3} = \cdot 9566$.

(5) $d = 0$, chance $= \frac{1}{2} + \frac{1}{2} = 1$.

The first and fourth of the above cases correspond with the first two mentioned in the question.

(ii.) Let $2d$ be greater than the coin's diameter. Here the chance will be found by multiplying the probability of the sphere's striking by the probability that, under these circumstances, the coin will strike also. That is, we have to multiply the chance of x being less than r by the chance that, x being known as less than r, the coin will strike. By the first of the above particular results, the chance is consequently seen to be for this case $\frac{r}{d} \cdot \frac{\pi}{4} = \frac{\pi r}{4d}$. Putting $d = \pi r$, this becomes $\frac{1}{4}$, which corresponds with the third case mentioned in the question.

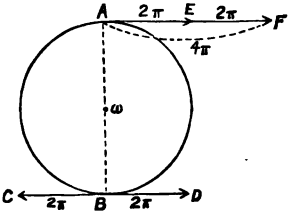
[The above solution gives for the first time, in a simple form, the long sought *general* formula for the probability. For other solutions see Vol. I., pp. 105—108 (2nd Edition), Vol. XXXIII., pp. 48—50, and Vol. XLV., p. 113. With respect to the last solution (by Mr. BIDDLE), which arrives at a different result, and on which criticism is invited, Mr. SIMMONS remarks as follows: — "The main fallacy in Mr. BIDDLE's solution clearly consists in the assumption that all the angles at which the plane of the coin may be inclined to the vertical plane passing through a wire are equally probable. For the most casual reader will easily perceive that the first plane admits of parallelism to the second in only one way, whereas it can be at right angles to it in an infinite number of ways. In other words, the probability that the planes make any given acute angle with each other *increases with the magnitude of the angle*."]

NOTE ON A FACT IN INSTANTANEOUS PHOTOGRAPHY.

By Professor EDMUND BORDAGE.

I have observed a curious fact in the experiments of Instantaneous Photography. In the photographic reproduction of a vehicle carried by a horse that trots, the wheels have more neatness in the inferior part than in the superior part.

The spokes of the wheels have evidently more velocity in the superior part than in the part adjacent to the ground.



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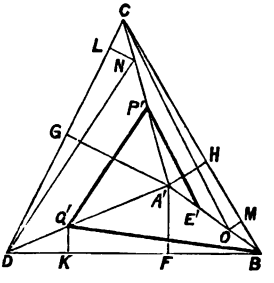
The geometric figure we have drawn gives the *mathematical* explanation of the fact.

Let us suppose the wheel is carried away (without turning on its axis) by the forces AE and BD equal to 2π (distance travelled by the vehicle in the unity of time.) But if, at the same time, the wheel begins to turn, the two ends A and B of the diameter AB shall be animated with a velocity EF equal to 2π, and with a velocity represented by BC and equal to EF, but having an opposed direction. The two velocities represented by BC and BD having the same point of application and two opposed directions destroy one another; the velocity in B is then null. To the point A, the two velocities represented by AE and EF, having the same direction, shall have $AF = 4\pi$ as resultant. The velocity represented by 4π is the double of the velocity of the vehicle. The contact point of the wheel with the ground is motionless in a given moment, while the other end A of the diameter is animated with a velocity that is the double of the vehicle-velocity.

8325. (By D. Biddle.)—A is the apex of a tetrahedron ABCD (regular or otherwise) which stands on a horizontal plane, and E is a point on the edge AB. A fly descends from E by making a circuit of the tetrahedron and finally arriving at B. Find at what points it must cross AC and AD, so as to maintain the same gradient to the horizon the whole way down.

Solution by the Proposer.

Let BDC represent the base of the tetrahedron, and A′ the foot of the perpendicular drawn from the apex. Join A′B, A′C, A′D, and let E′, P′, Q′ be the feet of perpendiculars from E, the starting point, and P, Q the required points. Join E′P′, P′Q′, Q′B; also draw CO, DN parallel to E′P′, P′Q′. Then (Euc. XI., 16), the vertical planes rising from DN, P′Q′ cut the face ACD in lines which are parallel to each other; and both have the same gradient, that is, by supposition, the gradient of BQ. Similarly, E′P′ and CO lie vertically underneath lines on the face ABC, which have the gradient of BQ.



Now the gradient of BQ = DK / DF . BQ′. Therefore

$$CL / CG . DN = DK / DF . BQ' = BM / BH . CO.$$

But $BQ' = \{(BD-DK)^2 + KQ'^2\}^{\frac{1}{2}}$, and $KQ' : A'F = DK : DF$,

therefore $$DK / DF . BQ' = 1 \Big/ \left(\frac{BK^2 . DF^2}{DK^2} + A'F^2\right)^{\frac{1}{2}}.$$

Consequently, if $CD = b$, $BD = c$, $BC = d$, $A'F = f$, $A'G = g$, $A'H = h$,

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DF = k, CG = l, BH = m, DK = x, CG = y, BH = z; then

$$\frac{(c-x)^2}{x^2}.k^2+f^2=\frac{(b-y)^2}{y^2}.l^2+g^2=\frac{(d-z)^2}{z^2}.m^2+h^2.$$

Also $$\frac{(k-x)(l-y)(m-z)}{k.l.m}=\frac{A'E'}{A'B}.$$

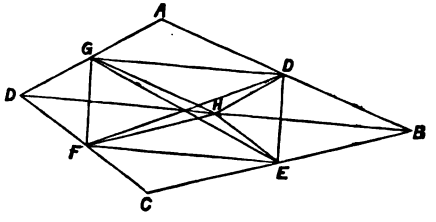
Therefore x, y, z can in any case easily be found. Where the tetrahedron is of the regular type, or where, the base being an equilateral triangle, the apex is anywhere vertically over the centroid of the base, $k=l=m$, and $x=y=z$. In these cases $\frac{(k-x)^3}{k^3}=\frac{A'E'}{A'B}$, whence $x=k\left\{1-\left(\frac{A'E'}{A'B}\right)^{\frac{1}{3}}\right\}$.

Consequently, in such cases, the length of the several edges from the apex to the base being regarded as unity, Q is distant from A the cube root of the distance AE, and P the square of the cube root. Thus, if AE = $\frac{8}{27}$AB, then AQ = $\frac{2}{3}$AD, and AP = $\frac{4}{9}$AC.

8512. (By Professor Bordage.)—Construct a quadrilateral, having given the mid-points of two opposites sides, the angle of intersection of the diagonals, the sum of the diagonals, the ratio of two opposites sides, and the sum of the squares of the other two sides.

Solution by J. Young, M.A.; D. Biddle; *and others.*

Suppose ABCD to be the required quadrilateral, and D, F the given mid-points; then the other two mid-points G, E are determined by the consideration that DG + GF is given, also ∠DGF; and similarly DE + EF and ∠DEF.



If H is the mid-point of DB, then DH, FH are the halves of AD, BC respectively, whose ratio is given; and, the line DF being given, a circular locus of H is given. Similarly the sum of the squares of GH, HE is given, and the base GE, thus a second circular locus for H, which determines it completely. To complete the construction, join H with D, G, F, E, and through G, D, E, F respectively draw parallels to the joining lines.

8587. (By Professor Haughton, F.R.S.)—Two equal masses of ice at 32° F. come into collision with equal and opposite velocities, and half

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their energy is converted into heat, the other half remaining as motion; find what their velocity must be so that the first half shall become steam at 212° F.

Solution by Rev. H. London, M.A.; Prof. Ignacio Beyens; *and others.*

Kinetic energy of system $= mv^2$ where m = mass of each portion.
Heat required to raise 1 kilo. of H_2O through 1° C. $= 424g$ (kilog-metre units).
Heat required for 1 gram. $= 42400g$ C. G. S. thermal units.
Therefore $\frac{1}{2}mv^2 = 716 \times 42400 \times 981m$ C. G. S. units.
Whence $v = 234920$ cents. per sec. $= 2{,}569$ yds. per sec. nearly.

8529. (By J. J. Walker, F.R.S.) — A homogeneous pyramid ABCDE stands upon a quadrilateral base, the diagonals BD, CE of which meet in O; show that the distance of its mass-centre from the face ABC is equal to 3 (Vol. ABCDE − Vol. ADEO) / 4 area ABC.

Solution by Rev. T. R. Terry, M.A.; A. Gordon; *and others.*

Let $OE = mCO$, $OD = nBO$, V = Vol. ABCDE, $\Delta = \Delta ABC$,

$$p = \text{perp. from O on ABC} = \frac{3V}{(1+m)(1+n)\Delta}.$$

Then for four tetrahedra which have A as vertex, the Vols. are proportional to 1, n, m, mn, and the distances of the mass-centres from ABC are $\frac{1}{4}p$, $\frac{1}{4}p\,(2+n)$, $\frac{1}{4}p\,(2+m)$, $\frac{1}{4}p\,(3+m+n)$, therefore distance for mass-centre of whole

$$= \tfrac{1}{4}p\,(1+m+n) = \frac{3V\,(1+m+n)}{4\,(1+m)\,(1+n)\,\Delta} = \frac{3\,(\text{Vol. ABCDE} - \text{Vol. ADEO})}{4\Delta}.$$

8553. (By Professor Mahendra Nath Ray, M.A., LL.B.)—If x, y, z denote the distances of the orthocentre from the angular points of the triangle ABC, prove that

$$xyz\,(ax+by+cz)^3 + abc\,(x^2+y^2+z^2)\,(ax+by+cz)^2 - 4a^3b^3c^3 = 0.$$

Solution by Rev. T. R. Terry, M.A.; Prof. Mathews, M.A.; *and others.*

Since $x = c\sin B - c\cos B\cot C = 2R\cos A = a\cot A$;

therefore $ax+by+cz = abcR^{-1}$.

Again, $x^2+y^2+z^2 = 4R^2(\cos^2 A + \cos^2 B + \cos^2 C)$,

therefore the required formula reduces to the well-known formula

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1.$$

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8392. (By Professor CATALAN.)—Démontrer que tout carré impair (supérieur à 9) argumenté de 5 est la somme de quatre carrés consécutifs ; et réciproquement.

Solution by W. J. BARTON, M.A.

We have $(2x+1)^2+5 = (x-1)^2+x^2+(x+1)^2+(x+2)^2$,
whence the result follows at once.

7191. (By Professor MINCHIN, M.A.)—In the general motion of any fluid, if, with reference to any rectangular axes, the components of distortion at any point are (a, b, c, s_1, s_2, s_3), and those of vortical spin are $(\omega_1, \omega_2, \omega_3)$; and if α is the normal acceleration at any point on any closed surface described in the fluid, prove that

$$\int \alpha d\mathrm{S} = \iiint (\theta^2 + \mathrm{D}_t\theta + 2\Omega^2 - 2\Sigma)\, dx\, dy\, dz,$$

where θ is the expansion, Ω the resultant vortical spin, Σ the strain invariant $ab+bc+ca-s_1^2-s_2^2-s_3^2$, and $\mathrm{D}_t\theta$ the total time-rate of increase of θ; the first integral being taken over the surface, and the second throughout its volume.

Solution by D. EDWARDES; SARAH MARKS; *and others.*

Let α, β, γ be the acceleration components, then the left side is

$$\int (l\alpha + m\beta + n\gamma)\, d\mathrm{S} = \iiint \left(\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz}\right) dx\, dy\, dz,$$

if l, m, n are the direction cosines of the outward drawn normal.

Now $$\alpha = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz},\ \beta = \&c.,\ \gamma = \&c.$$

From the first three terms in these we have $\frac{d\theta}{dt}$. The coefficient of u is

$$\frac{d^2u}{dx^2} + \frac{d^2v}{dy\,dx} + \frac{d^2w}{dz\,dx} \text{ or } \frac{d\theta}{dx};\ \text{ of } v,\ \frac{d\theta}{dy};\ \text{ and of } w,\ \frac{d\theta}{dz};$$

and the expression under the integral signs is therefore

$$\mathrm{D}_t\theta + \left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dy}\right)^2 + \left(\frac{dw}{dz}\right)^2 + 2\frac{dv}{dx}\frac{du}{dy} + 2\frac{dw}{dy}\frac{dv}{dz} + 2\frac{du}{dz}\frac{dw}{dx},$$

and $\frac{du}{dx} = a$, &c., $\frac{dv}{dx}\frac{du}{dy} = s_3^2 - w_3^2$, &c., and therefore, &c.

5178. (By Professor SYLVESTER, F.R.S.)—x, y, z are three given rational integral biquadratic functions of t; and two systems of ratios $a : b : c$, $\mathrm{A} : \mathrm{B} : \mathrm{C}$ are to be found such that $\frac{\mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2}{ax^2+by^2+cz^2}$ shall be an

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integral function of t; show that the problem admits in general of three, and never of more than three, solutions.

Solution by W. J. Curran Sharp, M.A.

Let $Ax^2 + By^2 + Cz^2 \equiv A_1 t^4 + 4B_1 t^3 + 6C_1 t^2 + 4D_1 t + E_1$, so that A_1, B_1, C_1, &c. are linear functions of A, B, and C, and

$$ax^2 + by^2 + cz^2 \equiv a_1 t^4 + 4b_1 t^3 + 3c_1 t^2 + 4d_1 t + e_1,$$

so that a_1, b_1, c_1, &c. are linear functions of a, b, and c, and let the quotient of the first expression divided by the second be $\psi(t)$, an integral function of t.

(1) If $\psi(t)$ be of the order 0 in $t = l$, say, the condition gives the equation $A_1 = la_1$, $B_1 = lb_1$, &c., or five equations of the form

$$\alpha A + \beta B + \gamma C = l(a_1 a + \beta_1 b + \gamma_1 c),$$

just enough to uniquely determine the ratios of

$$A : B : C : la : lb : lc, \quad \text{or of} \quad A : B : C \text{ and } a : b : c.$$

(2) If $\psi(t)$ be of the order 1 in $t = kt + l$, we have

$$a_1 = 0, \quad A_1 = 4kb_1, \quad 4B_1 = 4lb_1 + 6kc_1, \quad 6C_1 = 6lc_1 + 4kd_1,$$
$$4D_1 = 4la_1 + ke_1, \quad E_1 = le_1;$$

again just equations enough to uniquely determine

$$A : B : C \text{ and } a : b : c.$$

3. If $\psi(t)$ be of the order $2 = ht^2 + 2kt + l$, the same thing is true.

But if $\psi(t)$ be of a higher order in t, there are at least three equations $a_1 = 0$, $b_1 = 0$, $c_1 = 0$ involving the ratios $a : b : c$, and the problem of finding these is no longer capable of general solution.

8605. (By Rev. T. P. Kirkman, M.A., F.R.S.)—If, in any 4-edron $abcd$, ca be the 2-edral angle about ac, prove that

$$\cos \widehat{ca} + \cos(\widehat{cb} - \widehat{cd})\,[(bd)^2 - (cb - cd)^2]$$
$$= [\cos \widehat{ca} + \cos(\widehat{ad} - \widehat{ab})]\,[(bd)^2 - (ad - ab)^2].$$

Solution by the Proposer; Prof. Ignacio Beyens; *and others.*

By the spheric formula for A, when B and C are given,

$$\cos \widehat{ca} + \cos \widehat{cd} \cos \widehat{cb} = \cos bcd \sin \widehat{cd} \sin \widehat{cb},$$
$$\cos \widehat{ac} + \cos \widehat{ad} \cos \widehat{ab} = \cos bad \sin \widehat{ad} \sin \widehat{ab};$$

whence
$$\frac{\cos \widehat{ca} + \cos(\widehat{cd} - \widehat{cb})}{\cos \widehat{ac} + \cos(ad - ab)} = \frac{(1 + \cos bcd) \sin adb \,.\, \sin abd}{(1 + \cos bad) \sin cdb \,.\, \sin cbd}$$
$$= \frac{[(bc + cd)^2 - bd^2]\,\Delta^2 abd}{[(ba + ad)^2 - bd^2]\,\Delta^2 cbd} = \frac{bd^2 - (ba - da)^2}{bd^2 - (bc - dc)^2}.$$

Hence follows the pretty theorem (whose is it?) that

$$\widehat{ab} + \widehat{cb} = \widehat{ad} + \widehat{cd}, \text{ if } ab + cb = ad + cd.$$

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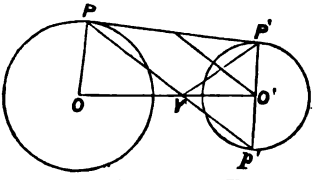
8451. (By Rev. J. J. MILNE, M.A. General case of Quest. 8266.) —If a centre of similitude of two circles be joined with the point of contact of one of the circles with either common tangent through the other centre of similitude, the line joining the middle point of the line so drawn and the centre of the circle bisects that common tangent.

Solution by R. F. DAVIS, M.A.; C. E. WILLIAMS, M.A.; *and others.*

Let PP′ be a common tangent to two circles, centres O, O′, so that OP, O′P′ are parallel. Then, if p' be the extremity of the diameter through P′, Pp' will pass through a centre of similitude Y. Hence, since O′P′ = O′p' a line through O′ parallel to Pp' will bisect both PP′ and P′Y.



[Quest. 8266 at once follows from the above: for, if AD meet EF in G, the triangles GAF, GDO are equal in all respects, therefore AF = DO = r. Similarly, from the other circle, $AF_1 = D_1O_1 = r_1$.]

8588. (By Professor HERMITE, LL.D.)—Déterminer la valeur de l'intégrale $J = \int_0^{\frac{1}{2}\pi} \cot(x+ai)\,dx$, a désignant une quantité réelle.

Solution by the PROPOSER.

On a : $\cot x = \dfrac{\cos x}{\sin x} = D_x \log \sin x$; l'intégrale indéfinie est par conséquent : $\log \sin(x+ai)$, soit $\sin(x+ai) = Re^{i\theta}$, R et θ étant des *quantités réelles*. Le module R est une fonction de x à sens unique, c'est la *quantité positive*, obtenue par la condition : $\sin(x+ai)\sin(x-ai) = R^2$. Pour $x = \frac{1}{2}\pi$ et $x = 0$ soit

$$\sin(\tfrac{1}{2}\pi+ai)\sin(\tfrac{1}{2}\pi-ai) = \cos(ai)\cos(-ai) = \cos^2 ai = R_1^2,$$

vous en concluez $R_1 = \cos ai$, quantité positive ; puis, pour $x = 0$,

$$\sin(ai)\sin(-ai) = R_0^2,$$

c'est-à-dire $R_0^2 = -\sin^2(ai)$; bref la partie réelle de J est

$$\log \frac{R_1}{R_0} = \tfrac{1}{2}\log \frac{R_1^2}{R_0} = \tfrac{1}{2}\log\{-\cot^2 ai\}.$$

Le point intéressant est la détermination précise de θ. Voici comment il faut opérer.

La relation que je viens d'employer, $\log \sin(x+ai) = \log R_1 + i\theta$, me donne, en prenant les dérivées par rapport à x,

$$\cot(x+ai) = D_x \log R + i\frac{d\theta}{dx}.$$

Cela étant, le premier membre peut s'écrire :

$$\cot(x+ai) = \frac{2\cos(x+ai)\sin(x-ai)}{2\sin(x+ai)\sin(x-ai)} = \frac{\sin 2x - \sin(2ai)}{2\,\mathrm{mod}^2 \sin(x+ai)}.$$

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Ayant séparé ainsi la partie réelle et la partie imaginaire, j'obtiens : $i\frac{d\theta}{dx} = -\frac{\sin(2ai)}{2\,\mathrm{mod}^2(x+ai)}$, ou plutôt, sous forme réelle :

$$\frac{d\theta}{dx} = \frac{i\sin(2ai)}{2\,\mathrm{mod}^2(x+ai)} = \frac{1}{2a}\left\{\frac{2ai\sin(2ai)}{\mathrm{mod}^2\sin(x+ai)}\right\}.$$

La quantité entre crochets est *essentiellement positive ;* donc l'angle θ croit avec x si a est >0, et décroit si a est <0. Or on a $e^{2i\theta} = \frac{\sin(x+ai)}{\sin(x-ai)}$. Pour $x=0$, $e^{2i\theta} = -1$, j'adopte la valeur $\theta = \frac{1}{2}\pi$; cela étant et x croissant d'une manière continue de $x=0$ à $x=\frac{1}{2}\pi$, θ varie ainsi d'une manière continue, et *en croissant*, si je suppose pour fixer les idées que a soit >0, jusqu'à la valeur θ', qui donne : $e^{2i\theta'} = \frac{\sin(\frac{1}{2}\pi+ai)}{\sin(\frac{1}{2}\pi-ai)} = 1$, par conséquent $\theta' = \pi$. La différence $\theta'-\theta = \frac{1}{2}\pi$ prouve que

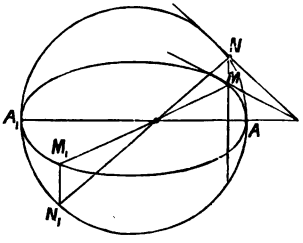
$$J = \tfrac{1}{2}\log[-\cot^2(ai)] + i\tfrac{1}{2}\pi,$$

pour $a>0$, etc.

8277. (By Professor Neuberg.)—Le cercle osculateur en un point M d'un ellipse rencontre cette courbe en M_1 ; le cercle osculateur en M_1 rencontre l'ellipse en M_2, etc. ; (1) La ligne brisée $MM_1, M_2\ldots$ peut-elle se fermer ? (2) Les cordes telles que MM_1, MM_2, $MM_3\ldots$ enveloppent des projections orthogonales d'épicycloïdes.

Solution by the Proposer.

1. Soient A, A′ les sommets de l'ellipse, et N, N_1, $N_2\ldots$ les points de la circonférence de diamètre AA′, qui correspondent aux points M, M_1, $M_2\ldots$ de l'ellipse. La tangente en M à l'ellipse et la corde d'osculation MM_1 ont des directions symétriques par rapport à la droite AA′ ; par suite, la tangente en N au cercle et la corde NN_1 font aussi des angles égaux, en sens contraires, avec AA′ ou ce qui revient au même, avec l'ordonnée NM. On conclut de là que les arcs AN_1, AN, comptés à partir de A dans des sens opposés, sont triple l'un de l'autre. Si l'arc $AN = a$, nous pourrons écrire en admettant des arcs négatifs $AN = a$, $AN_1 = -3a$, $AN_2 = 9a$, $AN_3 = -27a\ldots$, $AN_p = (-3)^p a$.



Pour que M_p coïncide avec M, il faut que l'on ait

$$(-3)^p a - a = 2\kappa\pi, \quad \frac{a}{2\pi} = \frac{2\kappa}{(-3)^p - 1}.$$

κ étant entier. Donc a doit être commensurable avec la circonférence, et si le rapport $a : 2\pi$ a pour expression la fraction irréductible $9/s$, s doit être premier avec 3. D'après le théorème de Fermat généralisé, cette condition est suffisante.

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Si $s = 3^r$, l'un des points N_1, N_2 ... coïncidera avec A ou A', et la série des points N s'arrête. Si $s = 3^r . \beta$, β étant un nombre premier avec 3, le point N_x coïncide avec l'un des suivants.

2. La corde NN_p se déplace dans le cercle de manière que les vitesses de ses extrémités N, N_p sont dans le rapport constant $1 : 3p$; donc elle enveloppe une épicycloïde. L'enveloppe de MM_p est donc une projection orthogonale de l'épicycloïde enveloppée par NN_p.

8593. (By Professor WOLSTENHOLME, M.A., Sc.D.)—The chord PQ of the surface $ax^2 + by^2 + cz^2 = 1$ (where $a + b + c = 0$) is normal at P; prove that PQ is the harmonic mean between the two principal radii of curvature at P.

Solution by Rev. T. R. TERRY, M.A.; Rev. H. LONDON, M.A.; *and others.*

Let X, Y, Z be the coordinates of P, and p = perpendicular from centre on tangent at P $= (a^2X^2 + \ldots)^{-\frac{1}{2}}$. Then the normal at P is

$$\frac{x-X}{aX} = \frac{y-Y}{bY} = \frac{z-Z}{cZ} = p \,.\, PQ;$$

therefore, substituting, $\frac{2}{PQ} = -p^3 (a^3X^2 + b^3Y^2 + c^3Z^2)$ ..... (1).

If u^{-1} be semi-axes of section by plane $aX + bY + cZ = 0$,

then $$\frac{a^2X^2}{a-u^2} + \ldots\ldots = 0;$$

therefore $$u^4 (a^2X^2 + \ldots) - u^2 \{(b+c)\, a^2X^2 + \ldots\} + \ldots\ldots = 0,$$

or, $$u^4 p^{-2} + u^2 (a^3X^2 + \ldots\ldots) + \ldots\ldots = 0,$$

$\therefore u_1^2 + u_2^2 = -p^2 (a^3X^2 + \ldots)$; $\therefore \frac{1}{R_1} + \frac{1}{R_2} = -p^3 (a^3X^2 + b^3Y^2 + c^3Z^2) = \frac{2}{PQ}$.

8584 & 8687. (8584). (By S. TEBAY, B.A.)—To draw a straight line which shall be trisected by two given circles.

(8687.) (By Rev. T. P. KIRKMAN, M.A., F.R.S.)—Given any two circles, meeting or not, by which a straight line L through neither centre is cut into three segments in order proportional to A, B, C. Required to draw a second straight line not equal to L, whose three segments so made shall be in the same proportion in the same order.

Solutions by (1) Rev. T. P. KIRKMAN, F.R.S.; (2) *the* PROPOSER.

1. Let it be required that the sections of the line be in proportion to three given numbers, K, L, M, *i.e.*, to k, 1, m. Taking for origin the mid-point of the middle section $2a$, and supposing the thing done, the

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circles will be, when asunder,

$$\{x+(k+1)a\}^2+(y-b)^2=r^2,\quad \{x-(m+1)a\}^2+(y-c)^2=p^2r^2\ldots(1,\ 2),$$

also $$a^2(2+k+m)^2+(c-b)^2=q^2r^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (3),$$

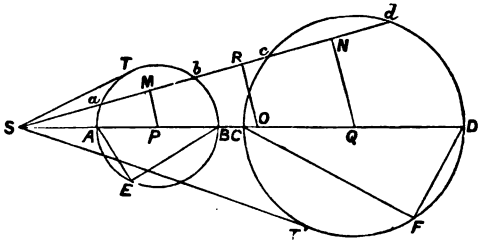
where qr is the distance between the centres. At $(-a,\ 0)$, (1) is $a^2k^2+b^2=r^2$, at $(a,\ 0)$, (2) is $a^2m^2+c^2=p^2(a^2k^2+b^2)$, and (3) is

$$a^2(2+k+m)^2+(c-b)^2=q^2(a^2k^2+l^2);$$

which is $\{(2+k+m)^2+(p^2k^2-m^2)\}\,a^2+p^2b^2-2b\{(p^2k^2-m^2)a^2+p^2b^2\}^{\frac{1}{2}}+b^2$ $=q^2(a^2k^2+b^2)$, giving $\quad A\dfrac{b^4}{a^4}+B\dfrac{b^2}{a^2}+C=0.$

If $2r\theta$ be the arc of the circle (1) cut off by the required line, $b:ak=\cot\theta$, wherefore θ, $r\cos\theta=b$, and $r\sin\theta=ak$, are given. To construct the solution, draw about the centre of (1) the circle with radius b, and also that with radius $\{b^2+(2a+ka)^2\}^{\frac{1}{2}}$, which latter must cut circle (2) in the point $(a,\ 0)$. The tangent drawn from $(a,\ 0)$ to the circle of radius b is the line required, and is cut by the two circles into three parts proportional to the given numbers K, L, M. When the solution is impossible the above quadratic will show it, with the reason why. With equal ease $b^2:a^2$ can be found from the proper (1), (2), (3), when one circle touches, meets, or contains the other. When the quadratic has two real roots, we get two values of θ and two of b, and two unequal lines, each of which solves the problem, and solves also Question 8687.

2. *Analysis.*—Let P, Q be the centres of the two circles, and *abcd* the required line, so that $ab=bc=cd$. Let ad meet PQ in S, the latter cutting the circles in A, B, C, D. Bisect PQ in O, and draw PM, QN,



OR perpendicular to ad. Also draw the tangents ST, ST′ to the circles P, Q. Then

$$Sa+Sb=2SM,\quad SM+SR=2Sb,\quad SM+SN=2SR,$$

therefore $\quad 2Sa=2SM-MR,\quad 2Sb=2SM+MR,$

therefore $\quad 4ST^2=4Sa\,.\,Sb=4SM^2-MR^2;$

also $\quad 4ST'^2=4Sc\,.\,Sd=4SN^2-MR^2,$

$$\therefore\ ST^2:ST'^2=4SM^2-MR^2:4SN^2-MR^2=4SP^2-PO^2:4SQ^2-PO^2.$$

But $\quad ST^2=SP^2-PA^2,\quad ST'^2=SQ^2-QC^2,$

therefore $\quad SP^2-PA^2:SQ^2-QC^2=SP^2-\frac{1}{4}PO^2:SQ^2-\frac{1}{4}PO^2$

$$=PA^2-\tfrac{1}{4}PO^2:QC^2-\tfrac{1}{4}PO^2,$$

therefore $\quad ST^2:ST'^2=4PA^2-PO^2:4QC^2-PO^2.$

Also, $\quad Sa:Sb=2SM-MR:2SM+MR=2SP-PO:2SQ-PO.$

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Whence the following *construction*:—In the circles P, Q apply the chords BE, DF each equal to PO, and join AE, CF. Find a point S in DA produced if necessary, such that tangents ST, ST′ being drawn to the circles P, Q, ST : ST′ :: AE : CF. Draw the secant S*ab*, such that

$$Sa : Sb :: 2SP - PO : 2SP + PO;$$

produce *ab* to cut the circle Q in *c*, *d*; then $ab = bc = cd$.

Demonstration:—Since

$$ST : ST' = AE : CF, \text{ therefore } ST^2 : ST'^2 = AE^2 : CF^2,$$

or $SP^2 - PA^2 : SQ^2 - QC^2 = AB^2 - PO^2 : CD^2 - PO^2$

$$= PA^2 - \tfrac{1}{4}PO^2 : QC^2 - \tfrac{1}{4}PO^2 = SP^2 - \tfrac{1}{4}PO^2 : SQ^2 - \tfrac{1}{4}PO^2,$$

therefore $ST^2 : ST'^2 = 4SP^2 - PO^2 : 4SQ^2 - PO^2$

$$= 4SM^2 - MR^2 : 4SN^2 - MR^2 = Sa \cdot Sb : Sc \cdot Sd.$$

But $Sa : Sb = 2SP - PO : 2SQ - PO = 2SM - MR : 2SN - MR,$

therefore $Sa : ab = 2SM - MR : 2MR$, $Sa : SM = 2SM - MR : 2SM,$

therefore $2Sa = 2SM - MR$, and $MR = ab$.

Again, $Sa^2 : Sa \cdot Sb = (2SM - MR)^2 : 4SM^2 - MR^2,$

therefore $Sa^2 : Sc \cdot Sd = (2SM - MR)^2 : 4SN^2 - MR^2.$

But $2Sa = 2SM - MR$, therefore $4Sc \cdot Sd = 4SN^2 - MR^2$;

and, since $Sc + Sd = 2SN$, therefore $Sd - Sc = MR = RN = ab$. Thus

$$ab = bc = cd.$$

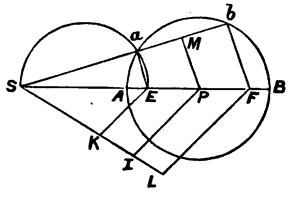
Scholium.—The construction here given depends upon the two following problems:—

I. To find a point in the straight line joining the centres of two given circles from which tangents drawn to them shall have a given ratio. This is a particular case of the more general problem of the circle of tangential ratio, which has been solved by SWALE and others (see *Diary* for 1855, p. 76).

II. From a given point to draw a straight line to cut a given circle so that the parts intercepted between the point and the circle shall have a given ratio.

Draw any straight line SL, and let SK be to SL in the given ratio. Bisect KL in I, and join IP; through K, L draw KE, LF parallel to IP; on SE describe a semicircle cutting the circle P in *a*; through *a* draw S*ab*; then

$$Sa : Sb = SK : SL.$$



Draw PM perpendicular to *ab*, and join *b*F. Because KI = IL, therefore EP = PF; and because the angles S*a*E, SMP are right angles, and *a*M = M*b*, therefore S*b*F is a right angle. Therefore SM : *a*M = SP : PE, and, dividendo et componendo, $Sa : Sb = SE : SF = SK : SL$. This problem seems to have been a stumbling-block to the present race of Lancashire Geometers. It was communicated to the proposer some years ago by Mr. William Smith, of Manchester, who had received it from Mr. Robert Howard, an old Diarian, formerly of Hooley Bridge. Its history has not been further traced.

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8555. (By Professor Genese, M.A.)—Any two circles are polar reciprocal with respect to four conics having their centres at the centres of similitude of the circles. The four conics are imaginary if the circles lie outside each other; two of the conics are real when one of the circles lies wholly within the other, and in that case their centre is the internal centre of similitude. Prove also that, if a and b be the usual semi-axes of one of the real conics, a is at right angles to the line of centres of the circles, and a^2/b is a mean proportional between the radii of the circles.

Solution by Professor Mathews, M.A.; Prof. Nash, M.A.; *and others.*

Let the circles be $x^2+y^2-\rho^2=0$, $(x-\delta)^2+y^2-\sigma^2=0$ .....(1, 2).

The polar of $(\rho\cos\theta,\ \rho\sin\theta)$ with regard to $\frac{(x-\xi)^2}{a^2}+\frac{y^2}{b^2}-1=0$, is

$$\frac{(\rho\cos\theta-\xi)(x-\xi)}{a^2}+\frac{\rho\sin\theta\,.\,y}{b^2}-1=0,$$ the envelope of which is

$$\frac{\rho^2(x-\xi)^2}{a^4}+\frac{\rho^2y^2}{b^4}-\left\{\frac{\xi(x-\xi)}{a^2}-1\right\}^2=0,$$

i.e., $$\frac{\rho^2-\xi^2}{a^4}x^2+\frac{\rho^2}{b^4}y^2-\frac{2\xi x(\rho^2-\xi^2-a^2)}{a^4}+\frac{\rho^2\xi^2-(\xi^2+a^2)^2}{a^4}=0 \quad .....(3).$$

Make this coincide with (2), then

$$\frac{\xi(\rho^2-\xi^2-a^2)}{\rho^2-\xi^2}=\delta,\quad \frac{\rho^2\xi^2-(\xi^2+a^2)^2}{\rho^2-\xi^2}=\delta^2-\sigma^2 \quad .....(\alpha,\ \beta),$$

therefore $$\rho^2\xi^2-(\delta^2-\sigma^2)(\rho^2-\xi^2)=\left\{\frac{\delta}{\xi}(\rho^2-\xi^2)-\rho^2\right\}^2$$

$$=\frac{\delta^2}{\xi^2}(\rho^2-\xi^2)^2-\frac{2\delta\rho^2}{\xi}(\rho^2-\xi^2)+\rho^4,$$

$$\rho^2-\xi^2=0,\ \text{or}\ \frac{\delta^2}{\xi^2}(\rho^2-\xi^2)-\frac{2\delta\rho^2}{\xi}+\delta^2-\sigma^2+\rho^2=0,$$

i.e., $$\frac{\delta^2\rho^2}{\xi^2}-\frac{2\delta\rho^2}{\xi}+\rho^2-\sigma^2=0,\quad \frac{\delta}{\xi}-1=\pm\frac{\sigma}{\rho},\quad \xi=\frac{\rho\delta}{\rho\pm\sigma},$$

i.e., the centre of the conic is at one of the centres of similitude.

Equation (α) gives

$$a^2=\left(1-\frac{\delta}{\xi}\right)(\rho^2-\xi^2)=\frac{\mp\sigma}{\rho}\cdot\frac{\{\delta^2-(\rho\pm\sigma)^2\}}{(\rho\pm\sigma)^2}\rho^2=\mp\rho\sigma\left\{\frac{\delta^2}{(\rho\pm\sigma)^2}-1\right\},$$

and again, comparing (3) with (2), $b^4=\frac{\rho^2}{\rho^2-\xi^2}a^4$.

For each value of ξ there is one value of a^2, and two values of b^2; so that there are four conics altogether. Finally,

$$\frac{b^4}{a^2}=\frac{\rho^2}{\rho^2-\xi^2}a^2=\rho^2\left(1-\frac{\delta}{\xi}\right)=\mp\rho^2\frac{\sigma}{\rho}=\mp\rho\sigma.$$

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8556. (By J. J. WALKER, M.A.)—If S denote the result of substituting the coordinates of a point P in the general equation of a central conic, prove that S is proportional to the square of the area of the quadrilateral formed by the tangents from P to the conic, and the central radii to the point of contact: viz., in Cartesian coordinates, $(\text{area})^2 = -\Delta S / C^2$.

Solutions by (1) *the* PROPOSER; (2) Rev. T. R. TERRY, M.A.

1. To prove the theorem for any system of trilinear coordinates, let (x, y, z) be P, O the centre, and PT, PT′ the tangents from P. Then, if $S \equiv ux + vy + wz$ (where $u \equiv ax + hy + gz$, $v \equiv hx + by + fz$, $w \equiv gx + fy + cz$), $A = bc - f^2 \ldots$, $H = fg - ch$, $\theta, \theta', \theta''$ the angles of the triangle of reference;

$$TT'^2 = -4\,(Au^2 + \ldots + 2Huv)\,(u^2 + \ldots - 2uv\cos\theta'')\,R^2 / (a\lambda^2 + \ldots + 2h\lambda\mu)^2,$$

where $\lambda = v\sin\theta'' - w\sin\theta' \ldots\ldots$ $R = \varpi\sin\theta + y\sin\theta' + z\sin\theta''$

(Proc. Lond. Math. Soc., VOL. IX., p. 232).

But $Au^2 + \ldots \equiv \Delta S$, $a\lambda^2 + \ldots \equiv (A\sin^2\theta + \ldots + 2H\sin\theta\sin\theta')\,S - \Delta R^2$;

viz., it may readily be verified that the coefficients of $\sin^2\theta$ and $\sin\theta'\sin\theta''$ in $a\lambda^2 + \ldots$ are $AS - \Delta x^2$ and $2(FS - \Delta yz)$ respectively. Again, if $\xi\eta\zeta$ are the coordinates of the centre, and ϕ the angle between OP and TT′,

$$\sin^2\phi = \left[\{v(x\eta - y\xi) - w(z\xi - x\zeta)\}\sin\theta + \ldots\right]^2 / R^2\,(u^2 + \ldots - 2uv\cos\theta'')\,OP^2$$

$$= \left[\{S\xi - x(u\xi + v\eta + w\zeta)\}\sin\theta + \ldots\right]^2 / \ldots$$

$$= \{S - x(a\xi + h\eta + g\zeta) - y(h\xi + \ldots) - z(g\xi + \ldots)\}^2 / (u^2 + \ldots - 2uv\cos\theta'')\,OP^2.$$

But $a\xi + h\eta + g\zeta = \Delta R\sin\theta / (A\sin^2\theta + \ldots + 2H\sin\theta\sin\theta')$,

$$h\xi + \ldots = \Delta R\sin\theta' / \ldots\; g\xi + \ldots = \Delta R\sin\theta'' / \ldots;$$

so that $\sin^2\phi = \{(A\sin^2\theta + \ldots)S - \Delta R^2\} / (A\sin^2\theta + \ldots)^2 (u^2 + \ldots - 2uv\cos\theta)\,OP^2$.

Hence, finally

$$\tfrac{1}{4}OP^2 \cdot TT'^2 \sin^2\phi = -\Delta R^2 S / (A\sin^2\theta + \ldots + 2F\sin\theta'\sin\theta'' + \ldots)^2.$$

2. It is clearly sufficient to prove theorem in case of the ellipse $x^2/a^2 + y^2/b^2 = 1$. In this case, if α and β are eccentric angles of points of contact, $\Delta = -1/a^2b^2$, $S = \tan^2\tfrac{1}{2}(\alpha - \beta)$, $C = 1/a^2b^2$, and it is easily proved that area $= ab\tan\tfrac{1}{2}(\alpha - \beta)$, whence the theorem follows.

[Mr. WALKER remarks that reference was made at the January Meeting of the London Mathematical Society to the property contained in this question, as having been published by him in the *Educational Times* many years ago. No solution, however, had been offered previous to this reproduction of the Question. At p. 39 of Vol. XIV., Mr. WALKER has shown that S is also proportional to the product of the four normals divided by that of the two focal distances.]

8609. (By R. TUCKER, M.A.)—Prove that

$$\begin{vmatrix} -b^2c^2, & ab(a^2+c^2), & ac(a^2+b^2) \\ ab(b^2+c^2), & -c^2a^2, & bc(a^2+b^2) \\ ac(b^2+c^2), & bc(a^2+c^2), & -a^2b^2 \end{vmatrix} = (a^2b^2 + b^2c^2 + c^2a^2)^3.$$

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Solution by Prof. IGNACIO BEYENS; ISABEL MADISON; *and others.*

Ajoutons nous à la (1) ligne de la déterminant le produit de la (2) par $-a/c$ et à la (2) le produit à la (3) par $-b/c$ et nous aurons en appelant

$$a^2b^2+b^2c^2+c^2a^2 = S \begin{vmatrix} -S, & 0, & a/c\,S \\ 0, & -S, & b/c\,S \\ ac(b^2+c^2), & bc(a^2+c^2), & -a^2b^2 \end{vmatrix}.$$

Ajoutons à la (3) colonne la (1) multipliée par a/c et la (2) multipliée par b/c et nous obtiendrons

$$\begin{vmatrix} -S, & 0, & 0 \\ 0, & -S, & 0 \\ ac(b^2+c^2), & bc(a^2+c^2), & S \end{vmatrix} = S^3, \text{ ou } = (a^2b^2+a^2c^2+b^2c^2)^3.$$

8539. (By P. C. WARD, M.A.) — If $p\cos\frac{1}{2}(\alpha+\beta)+q\sin\frac{1}{2}(\alpha+\beta) = k\cos\frac{1}{2}(\alpha-\beta)$; show that

$$\frac{d\alpha}{[p^2+q^2+k^2-2pk\cos\alpha-2qk\sin\alpha]^{\frac{1}{2}}} = \frac{d\beta}{[p^2+q^2+k^2-2pk\cos\beta-2qk\sin\beta]^{\frac{1}{2}}}.$$

Solution by D. EDWARDES; Professor IGNACIO BEYENS; *and others.*

Let p, q be the Cartesian coordinates of a fixed point O within a circle of radius k, and α, β the angular coordinates of the extremities of a chord POQ. Then $OP = \{p^2+q^2+k^2-2ph\cos\alpha-2qh\sin\alpha\}^{\frac{1}{2}}$, OQ = &c. If P′OQ′ be a near position of the chord,

$$\frac{PP'}{OP} = \frac{QQ'}{OQ} \quad \text{or} \quad \frac{d\alpha}{OP} = \frac{d\beta}{OQ}.$$

8608. (By S. ROBERTS, M.A.)—Show that, if a given triangle ABC be taken as the triangle of reference, and straight lines be drawn through any point on the line $\alpha\cos A+\beta\cos B+\gamma\cos C = 0$, parallel to the sides of the given triangle, the six finite intersections with the sides lie on a rectangular hyperbola.

Solution by the PROPOSER.

Denoting the three right lines by

$$x\cos\alpha+y\sin\alpha = x\cos\beta+y\sin\beta = x\cos\gamma+y\sin\gamma = 0,$$

and the sides of the given triangle by

$$x\cos\alpha+y\sin\alpha-a = x\cos\beta+y\sin\beta-\beta = x\cos\gamma+y\sin\gamma-\gamma = 0,$$

the six finite intersections lie on the conic

$$MN\alpha+NL\beta+LM\gamma-L\beta\gamma-M\gamma\alpha-N\alpha\beta+\alpha\beta\gamma = 0,$$

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where L, M, N are written for

$$x\cos\alpha + y\sin\alpha, \quad x\cos\beta + y\sin\beta, \quad \text{and} \quad x\cos\gamma + y\sin\gamma.$$

In order that this conic may be a rectangular hyperbola, we must have

$$\alpha\cos(\beta-\gamma) + \beta\cos(\gamma-\alpha) + \gamma\cos(\alpha-\beta) = 0,$$

or

$$\alpha\cos A + \beta\cos B + \gamma\cos C = 0,$$

representing a line of known geometrical relations.

8541. (By C. E. McVicker, M.A.)—Prove that the normal at any point P on a parabola cuts off from the curve an area equal to eight times the area cut off by the focal chord drawn through P.

Solution by Rev. T. R. Terry, M.A.; G. G. Storr, B.A.; *and others.*

Let coordinates of P be $(am^2, 2am)$, and let T be pole of focal chord PQ and T′ pole of normal chord PR; then

$$PQ = \frac{a(1+m^2)^2}{m^2}, \quad \text{perpendicular from } T = \frac{a(1+m^2)}{m},$$

therefore

$$\Delta TPQ = \frac{a^2(1+m^2)^3}{2m^3};$$

$$PR = \frac{4a(1+m^2)^{\frac{1}{2}}}{m^2}, \quad \text{perpendicular from } T = \frac{2a(1+m^2)^{\frac{1}{2}}}{m},$$

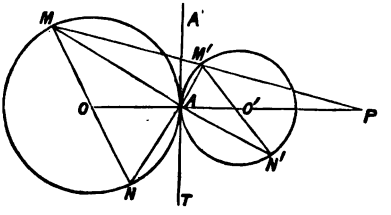
therefore

$$\Delta T'PR = \frac{4a^2(1+m^2)^3}{m^3};$$

therefore $\Delta T'PR = 8\Delta TPQ$, whence, &c.

8614. (By Maurice D'Ocagne.)—Les cercles C et C′, dont les centres sont O et O′, sont tangents en A; soient M un point du cercle C, M′ un point du cercle C′, tels que l'angle MAM′ soit droit. La droite MM′ passe par un point fixe qui est le conjugué harmonique du point A par rapport aux points O et O′.

Solution by Professors Ignacio Beyens *and* Matz.



Nous aurons dans la figure menant la tangente TA′ angle MAA′ = MNA, MAA′ = TAN′ = AM′N′ et par suite MNA = AM′N′, MO, M′O′ sont des rayons parallèles et on déduit

$$\frac{MO}{M'O} = \frac{PO}{PO} \text{ ou } \frac{AO}{AO'} = \frac{PO}{PO'}$$

(C. Q. F. D.)

(MN et M′N′ passent par O et O′ parce que l'angle MAM′ étant droit aussi le seront MAN, M′AN′.)

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8601. (By Professor Byomakesa Chakravarti, M.A.)—A lamina bounded by a cycloid and its base has its centre of gravity at the middle point of its axis. It is placed with its base vertical on a perfectly rough horizontal plane, and allowed to roll down: show that, at the moment its vertex reaches the ground, its angular velocity is $\left\{\frac{2ag(\pi-1)}{a^2+k^2}\right\}^{\frac{1}{2}}$, where a is the radius of the generating circle, and k the radius of gyration about the centre of gravity.

Solution by Rev. T. R. Terry, M.A.; Professor Matz, M.A.; *and others.*

The potential energy lost is $Mga(\pi - 1)$, and the kinetic energy gained is $\frac{1}{2}M(a^2+k^2)\omega^2$, whence equating we get the value of ω.

8606. (By Asûtosh Mukhopâdhyây, M.A., F.R.A.S.)—Two sources of heat are placed at the foci of an ellipse, the strength of one being m times that of the other. Show that the points on the periphery at which a person would feel the maximum quantity of heat are determined as the intersection of the ellipse with the line $x = \frac{a}{e} \cdot \frac{1-m^{\frac{1}{2}}}{1+m^{\frac{1}{2}}}$.

Solution by Rev. T. R. Terry, M.A.; Professor Beyens; *and others.*

The flow being supposed to take place in two dimensions, we have to make $\frac{1}{r^2} + \frac{m}{r'^2} = max$ while $r + r' = 2a$, whence $\frac{1}{r^3} = \frac{m}{r'^3}$ or $\frac{r-r'}{r+r'} = \frac{1-m^{\frac{1}{2}}}{1+m^{\frac{1}{2}}}$, therefore

$$x = \frac{a}{e} \cdot \frac{1-m^{\frac{1}{2}}}{1+m^{\frac{1}{2}}}.$$

6317. (By Dr. Hopkinson, F.R.S.)—A viscous fluid occupies the space between two infinite parallel planes, whereof one is fixed, whilst the other moves parallel to itself with simple harmonic motion $A\cos nt$. Show that the consequent tangential force on the fixed plane has a maximum value per unit of area $= 2A\lambda\mu n(\cosh 2\lambda l - \cos 2\lambda l)^{-\frac{1}{2}}$, where l is the distance between the planes, $\lambda^2 = \frac{\rho n}{2\mu}$, and the fluid in contact with a plane is assumed to move with the plane.

Solution by D. Edwardes.

Taking the origin in the fixed plane, the x-axis in the direction of the vibration, and y-axis perpendicular to the planes, the equations of motion

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reduce to $\frac{du}{dt} = -\frac{1}{\rho}\frac{dp}{dx} - \mu'\frac{d^2u}{dy^2}$, $\frac{dp}{dy} = 0$, $\frac{dp}{dz} = 0$ where $\mu' = \frac{\mu}{\rho}$. Hence $\frac{du}{dt} - \mu'\frac{d^2u}{dy^2} = -\frac{1}{\rho}\frac{dp}{dx}$ = a function of t only $= f(t)$.

Assume $$u = \mathrm{P}\sin nt + \mathrm{Q}\cos nt,$$

whence the equation is satisfied by

$$f(t) = 0, \quad \mu'\frac{d^2\mathrm{P}}{dy^2} + n\mathrm{Q} = 0, \quad \mu'\frac{d^2\mathrm{Q}}{dy^2} - n\mathrm{P} = 0,$$

and we easily get

$$u = \left\{e^{\lambda y}(c_1\cos\lambda y + c_2\sin\lambda y) + e^{-\lambda y}(c_3\sin\lambda y + c_4\cos\lambda y)\right\}\sin nt$$
$$+ \left\{e^{\lambda y}(c_1\sin\lambda y - c_2\cos\lambda y) + e^{-\lambda y}(c_3\cos\lambda y - c_4\sin\lambda y)\right\}\cos nt.$$

But $u = 0$ when $y = 0$ for all values of t, whence $c_3 = -c_1$, and $c_4 = c_2$ and

$$u = \left\{2c_1\cos\lambda y\sinh\lambda y + 2c_2\sin\lambda y\cosh\lambda y\right\}\sin nt$$
$$+ \left\{2c_1\sin\lambda y\cosh\lambda y - 2c_2\cos\lambda y\sinh\lambda y\right\}\cos nt.$$

Also $u = -n\mathrm{A}\sin nt$ when $y = l$, whence

$$c_1\sin\lambda l\cosh\lambda l - c_2\cos\lambda l\sinh\lambda l = 0,$$
$$-n\mathrm{A} = 2c_1\cos\lambda l\sinh\lambda l + 2c_2\sin\lambda l\cosh\lambda l,$$

which determine c_1 and c_2. Now the stress on the fixed plane

$$= \mu\left(\frac{du}{dy}\right)_{y=0} = 2\lambda\mu\left\{(c_1 + c_2)\sin nt + (c_1 - c_2)\cos nt\right\},$$

therefore maximum stress $= 2\lambda\mu(2c_1^2 + 2c_2^2)^{\frac{1}{2}}$

$$= \frac{2\lambda\mu n\mathrm{A}}{(2\cos^2\lambda l\sinh^2\lambda l + 2\sin^2\lambda l\cosh^2\lambda l)^{\frac{1}{2}}} = \frac{2\lambda\mu n\mathrm{A}}{(\cosh 2\lambda l - \cos 2\lambda l)^{\frac{1}{2}}}.$$

8596. (By Professor Hadamard. Généralisation de la Quest. 8510.) —Trouver (1) les polynômes $\mathrm{F}(x)$, de degré $2p$, tels que

$$(c + bx)^{2p}\,\mathrm{F}\left(\frac{a - cx}{c + bx}\right) \equiv (c^2 + ab)^p\,\mathrm{F}(x),$$

et

$$(c' + b'x)^{2p}\,\mathrm{F}\left(\frac{a' - c'x}{c' + b'x}\right) \equiv (c'^2 + a'b')^p\,\mathrm{F}(x).$$

En général, ces polynômes sont de la forme $(\mathrm{A}x^2 + \mathrm{B}x + \mathrm{C})_p$, A, B, C étant des constantes déterminées. Examiner (2) le cas (réalisé dans la question 8510) où il en est autrement, et voir comment doivent être choisies à cet effet les constantes a, b, c, a', b', c'.

Solution by the Proposer.

Soit x_0 une des racines de l'équation $\mathrm{F}(x) = 0$, il lui correspondra une racine $x_1 = \frac{a - cx_0}{c + bx_0}$, puis une racine $x_2 = \frac{a' - c'x_1}{c' + b'x_1}$ ..... (1, 2), et ainsi de suite. Il faut que cette suite se ferme, sans quoi l'équation

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aurait une infinité de racines. On a donc $x_{2k-1} = \frac{a' - c'x_0}{c' + b'x_0}$ et $x_{2k} = x_0$. Mais d'autre part les équations (1), (2) et les suivantes donnent $x_{2k} = \frac{A + Bx_0}{C + Dx_0}$. L'équation $x_{2k} = x_0$ est donc du 2^e degré en x_0. Or les $2k$ termes de la suite satisfont à cette équation. Donc $k = 1$, à moins que l'équation ne soit identique. Si $k = 1$, comme on a

$$x_2 = \frac{ca' - ac' + (cc' + ba')\,x_0}{cc' + ab' + (bc' - cb')\,x_0} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

x_0 est racine de l'équation

$$P \equiv (bc' - cb')\,x^2 + (ab' - ba')\,x + ac' - ca' = 0 \quad \ldots\ldots\ldots\ldots (4).$$

Si donc l'équation $x_{2k} = x_0$ n'est identique pour aucune valeur de k la forme générale des polynômes est P^p.

Soient m et n les racines de l'équation (4). L'équation (3) peut s'écrire,

si m et n sont distincts, $\frac{x_2 - m}{x_2 - n} = \alpha \frac{x_0 - m}{x_0 - n}$ ……(5),

et s'ils ne le sont pas $\frac{1}{x_2 - m} - \frac{1}{x_0 - m} = c.$

Dans ce dernier cas la suite ne peut évidemment se fermer. Dans le premier il faut et il suffit que α soit racine $k^{ième}$ de 1. Alors l'équation qui a pour racines $x_0, x_2, \ldots x_{2k-2}$ est

$$\left(\frac{x - m}{x_0 - m}\right)^k - \left(\frac{x - n}{x_0 - n}\right)^k = 0.$$

Et celle qui a pour racines $x_1, x_3 \ldots, 2_{2k-1}$ s'obtient en y remplaçant x par $\frac{a - cx}{c + bx}$. Mais $\left(\frac{a - cx}{c + bx} - m\right) \Big/ \left(\frac{a - cx}{c + bx} - n\right) = \frac{a - cm - (c + bm)\,x}{a - cn - (c + bn)\,x}$ ……(6).

Or on a $a - cm = (c + bm)\,n$. Car cette égalité peut s'écrire

$$bmn + c\,(m + n) - a = 0,$$

et cette dernière se vérifie immédiatement en remplaçant $m + n$ et mn par leurs valeurs tirées de l'équation (4). Donc la fraction (6) peut s'écrire $\frac{c + bm}{c + bn} \cdot \frac{x - n}{x - m}$. L'équation qui a pour racines $x_1, x_3 \ldots x_{2k-1}$ est donc

$$\left\{\frac{(c + bm)(x - n)}{x_0 - m}\right\}^k - \left\{\frac{(c + bn)(x - m)}{x_0 - m}\right\}^k = 0.$$

L'ensemble donne $Q + \lambda P^m = 0$. En posant

$$Q = \left\{(c + bm)(x - n)^2\right\}^k + \left\{(c + bn)(x - m)^2\right\}^k.$$

Et par suite la forme générale demandée est

$$P^h \left\{A P^{mg} + A_1 P^{m(g-1)} Q + \ldots + A_i P^{m(g-i)} Q^i + \ldots + A_g Q^g\right\}.$$

8583. (By J. Brill, M.A.)—If p denote the pressure at any point in a homogeneous incompressible fluid which is in irrotational motion of a two-dimensional character, subject to the action of a field of force whose

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potential satisfies LAPLACE's equation, prove that

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\left\{\log\left(\frac{d^2p}{dx^2}+\frac{d^2p}{dy^2}\right)\right\}=0.$$

Solution by D. EDWARDES; Prof. SIRCOM, M.A.; *and others.*

If ϕ be the velocity potential, so that

$$\frac{d\phi}{dx}=u,\ \frac{d\phi}{dy}=v,\ \frac{p}{\rho}+\frac{d\phi}{dt}+\tfrac{1}{2}(u^2+v^2)+V=\text{constant},$$

and therefore $\frac{1}{\rho}\nabla^2 p+\tfrac{1}{2}\nabla^2 q^2=0$, where $q^2=u^2+v^2$.

But $$\frac{du}{dx}+\frac{dv}{dy}=0, \text{ and } \frac{du}{dy}-\frac{dv}{dx}=0,$$

whence $$\tfrac{1}{2}\nabla^2 q^2=2\left\{\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2\right\},$$

and the result follows at once by differentiating, observing that

$$\nabla^2 u=0,\ \nabla^2 v=0.$$

8568. (By Rev. T. R. TERRY, M.A.)—Three equal circles (radius a) pass through a point O and intersect two and two in the points A, B, C. Three circles are drawn each passing through O and one of the points A, B, C, and cutting the third circle orthogonally in the points D, E, F. Show that (1) the circles AD, BE, CF have a common radical axis; and (2) the circle drawn through the points D, E, F touches the circle whose centre is O and radius $2a$, and which touches the first three circles.

Solution by the PROPOSER; Prof. IGNACIO BEYENS; *and others.*

In a plane triangle ABC it is well known (1) that three perpendiculars AD, BE, CF meet in a point; and (2) the circle through the points D, E, F touches the inscribed circle. Invert the whole figure with respect to the centre of the inscribed circle, and we obtain the given theorems.

8465 & 8500. (By D. EDWARDES.)—(8465) If

$$S_n=1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+1/n,$$

prove that

$$S_1-\left(\frac{1}{2}\right)^2\frac{S_2}{2}+\left(\frac{1.3}{2.4}\right)^2\frac{S_3}{3}-\left(\frac{1.3.5}{2.4.6}\right)^2\frac{S_4}{4}+\ldots=\frac{16}{\pi}(1-\log_e 2).$$

(8500) Prove that $$\int_0^1\int_0^{\frac{1}{2}\pi}\frac{\log(1-k)}{(1-k\sin^2\phi)^{\frac{1}{2}}}\,dk\,d\phi=8\log 2-8.$$

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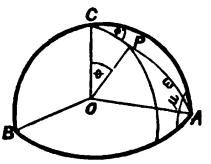
Solution by the PROPOSER.

Let θ, ϕ be the angular coordinates of a point P on a sphere of unit radius, and ω, μ the angular coordinates of the same point referred to A.



Then
$$\sin\theta\cos\phi = \cos\omega,$$
$$\cos\theta = \sin\omega\sin\mu,$$
$$\sin\theta\sin\phi = \sin\omega\cos\mu.$$

Hence
$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\sin\theta\log(\sin\theta\cos\phi)\,d\theta\,d\phi = \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\sin\omega\log(\cos\omega)\,d\omega\,d\mu$$
$$= \tfrac{1}{2}\pi\int_0^{\frac{1}{2}\pi}\sin\omega\log(\cos\omega)\,d\omega = -\tfrac{1}{2}\pi,$$

so that
$$\int_0^{\frac{1}{2}\pi}\sin\theta\log\sin\theta\,d\theta + \int_0^{\frac{1}{2}\pi}\log(\cos\phi)\,d\phi = -\tfrac{1}{2}\pi,$$

whence
$$\int_0^{\frac{1}{2}\pi}\sin\theta\log(\sin\theta)\,d\theta = \log 2 - 1.$$

Again,
$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\sin\theta\cos\phi\log(\sin\theta\cos\phi)\,d\theta\,d\phi$$
$$\left(= 2\int_0^{\frac{1}{2}\pi}\sin\theta\log\sin\theta\,d\theta\right) = \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{\sin\omega\cos\omega\log\cos\omega}{(1-\sin^2\mu\sin^2\omega)^{\frac{1}{2}}}\,d\omega\,d\mu,$$

so that
$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{\sin\omega\cos\omega\log\cos\omega}{(1-\sin^2\mu\sin^2\omega)^{\frac{1}{2}}}\,d\omega\,d\mu = 2\log 2 - 2;$$

or, as it may be written,
$$\int_0^1\int_0^{\frac{1}{2}\pi}\frac{\log(1-k)}{(1-k\sin^2\phi)^{\frac{1}{2}}}\,d\theta\,d\phi = 8\log 2 - 8.$$

Now
$$\int_0^1 k^n\log(1-h)\,dk = \int_0^1(1-k)^n\log k\,dh,$$

and since
$$\int_0^1 k^r\log k\,dk = -\frac{1}{(r+1)^2},$$

we have
$$\int_0^1 k^n\log(1-h)\,dk = -\left(\frac{1}{1^2} - n\frac{1}{2^2} + \frac{n(n-1)}{2}\cdot\frac{1}{3^2}\cdots\right)$$
$$= -\frac{1}{n+1}\left(P_1 - \frac{P_2}{2} + \frac{P_3}{3} - \ldots\right),$$

where P_r = coefficient of x^r in the expansion of $(1+x)^{n+1}$. But the series within the brackets $= 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n+1}$ (see *Reprint*, Vol. XXXV., p. 49) $= S_{n+1}$ say, therefore
$$\int_0^1 k^n\log(1-h)\,dk = -\frac{S_{n+1}}{n+1}.$$

Also
$$\int_0^{\frac{1}{2}\pi}\sin^{2m}\phi\,d\phi = \frac{1\cdot3\cdot5\ldots2m-1}{2\cdot4\cdot6\ldots2m}\cdot\frac{\pi}{2}.$$

Hence, expanding $(1-k\sin^2\phi)^{-\frac{1}{2}}$, and integrating, we have
$$S_1 - \left(\frac{1}{2}\right)^2\frac{S_2}{2} + \left(\frac{1.3}{2.4}\right)^2\frac{S_3}{3} - \&c. = -\frac{2}{\pi}\int_0^1\int_0^{\frac{1}{2}\pi}\frac{\log(1-k)}{(1-k\sin^2\phi)^{\frac{1}{2}}}\,dk\,d\phi$$
$$= \frac{16}{\pi}(1-\log 2).$$

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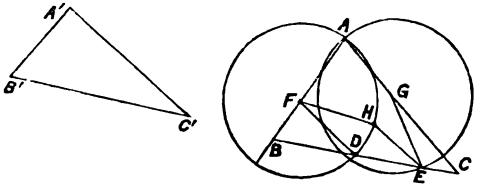
8590. (By Professor BORDAGE.)—An isosceles triangle ABC being given, find in its base BC two points D and E, and in the sides AB, AC two points F, G, such that the pentagon AFDEG shall have equal sides.

Solution by Rev. JAMES WHITE, M.A.

Let A'B'C' be any triangle, take an angle BAC = the angle B'A'C', in the sides take the points F and G equidistant from A. Describe two circles

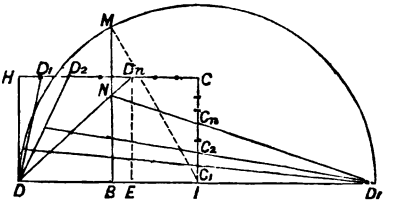


with F and G as centres passing through A. Draw a radius FH making the angle AFH = the angle A'B'C'. From H inflect a line HE to the circumference of the circle whose centre is G = radius of circle. Draw FD parallel to HE; join DE, this is obviously parallel and equal to FH. Join GE. Now, if DE be produced to meet AF and AG in B and C, the triangle ABC is equiangular to A'B'C', and has the pentagon AFDGE equilateral. The pentagon AFDEG is evidently given in form, and can easily be constructed. This gives the angles BAD and CAE, and determines the points D and E, and the length of the sides. The problem is thus extended to any triangle.

A NEW CONSTRUCTION BY POINTS OF THE ELLIPSIS.

By Professor EDMUND BORDAGE.

Let $DD' = ea$ be the major axis of the ellipse, and $IC = b$ the semi-minor axis. Let us construct the rectangle DICH on the half-axis ID, IC, and divide the sides into one part more than will suffice to furnish points belonging to the quarter of the ellipse. The number of these divisions is m. Let us call the points of division D_1, D_2, D_{m-1}; C_1, C_2, C_{m-1}. I say that the straight lines DD_1, $D'C_1$; DD_2, $D'C_2$; ... DD_n, $D'C_n$, intersect two and two through points lying on the ellipse.



Indeed, let us consider the lines DD_n, $D'C_n$ that intersect through N. Let NB be the ordinate of the point N; its prolongation cuts in M the half-circumference. We must prove that $\frac{NB}{MB} = \frac{b}{a}$;

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but we have $\frac{NB}{D_nE} = \frac{DB}{DE}$; and $D_nE = b$, $DE = \frac{n}{m} . a$;

then $NB = \frac{b}{a} . \frac{m}{n} . DB$.....(1).

We also have $\frac{NB}{C_nI} = \frac{D'B}{D'I}$, but $C_nI = \frac{n}{m} . b$, $D'I = a$;

then $NB = \frac{b}{a} . \frac{n}{m} . D'B$ .....(2).

If we multiply (1) and (2), we have

$$(NB)^2 = \frac{b^2}{a^2} . DB . D'B = \frac{b^2}{a^2} (a-IB)(a+IB) = \frac{b^2}{a^2} [a^2-(IB)^2].$$

But $a^2-(IB)^2 = (IM)^2-(IB)^2 = (MB)^2$,

then $(NB)^2 = \frac{b^2}{a^2} . (MB)^2$ and $\frac{NB}{QB} = \frac{b}{a}$. (C. Q. F. D.)

8514. (By Professor Mahendra Nath Ray, M.A., LL.B.)—Find the values of x, y, z, u, v, w from the following equations:—

$$yz-u^2 = a^2,\ zx-v^2 = b^2,\ xy-w^2 = c^2,\ vw-xu = d^2,$$
$$wu-yv = e^2,\ uv-zw = f^2.$$

Solution by Professors Sircom and Beyens *and* Isabel Madison.

We have $\Delta^2 = \begin{vmatrix} a^2, & f^2, & e^2 \\ f^2, & b^2, & d^2 \\ e^2, & d^2, & c^2 \end{vmatrix} = \begin{vmatrix} x, & w, & v \\ w, & y, & u \\ v, & u, & z \end{vmatrix}^2$,

since the first determinant is the reciprocal of the second; and for the same reason, x, y, z, &c., are proportional to the minors of the first determinant, hence $x = (b^2c^2-d^4) / \Delta$, and so for the rest.

8595. (By Professor Hudson, M.A.)—Prove that

$$\int_0^{\pi} (A_0 + A_1 \cos\theta + A_2 \cos^2\theta + \dots A_n \cos^n\theta + \dots)\, d\theta$$
$$= \pi \Sigma A_{2n} \frac{(2n-1)(2n-3)\dots 1}{2n(2n-2)\dots 2}.$$

Solution by Rev. T. R. Terry, M.A.; R. Knowles, B.A.; *and others.*

The required result follows at once from the two well-known theorems

$$\int_0^{2a} \phi(x)\, dx = \int_0^{a} \{\phi(x) + \phi(2a-x)\}\, dx \quad \dots\dots(1),$$

$$\int_0^{\frac{1}{2}\pi} \cos^{2n}\theta\, d\vartheta = \frac{2n-1}{2n} \int_0^{\frac{1}{2}\pi} \cos^{2n-2}\theta\, d\theta \quad \dots\dots(2).$$

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7434. (By the Editor.)—Trace the locus of the feet of the normals drawn from a given point on the conics inscribed in a given square, distinguishing the various forms of the curve for different positions of the given point.

Solution by Professor Nash, M.A.

If the sides of the square are $x = \pm a$, $y = \pm a$, the equation of an inscribed conic is $x^2 + 2\lambda xy + y^2 = a^2(1-\lambda^2)$ .....(1).

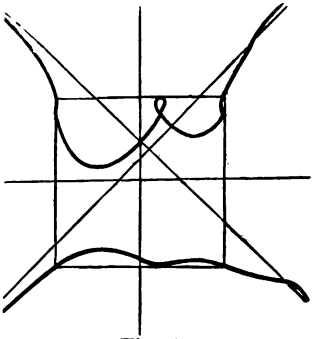

Fig. 1.

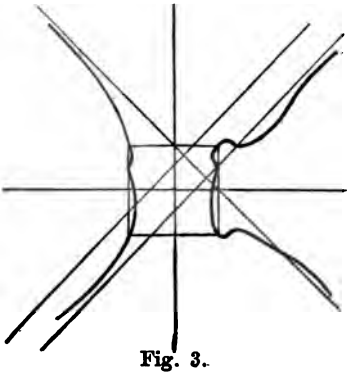

Fig. 3.

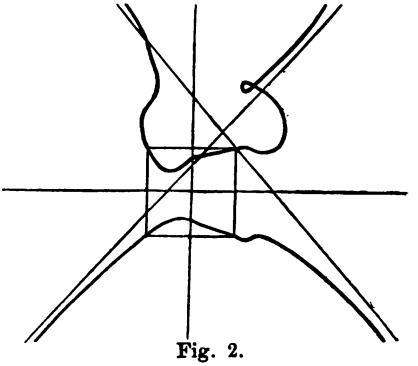

Fig. 2.

The equation of the normal at X, Y is

$$(x-X)(\lambda x+y) = (y-Y)(X+\lambda Y).$$

Interchanging (x, y), (X, Y) to denote that the normal passes through the fixed point XY, we get the locus of the foot of the normal by eliminating λ between (1) and $(X-x)(\lambda x+y) = (Y-y)(x+\lambda y)$.

Performing the elimination, we find that the locus consists of the diagonals of the square and the curve $(x^2-a^2)(x-X)^2 = (y^2-a^2)(y-Y)^2$;

the asymptotes of this curve are $x \pm y = \frac{1}{2}(X \pm Y)$.

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The curve passes through the nine points of intersection of the lines $x = \pm a$, $x = X$, with the lines $y = \pm a$, $y = Y$, the point X, Y being a double point, the tangents at which make angles with the axis of $x = \sin^{-1}\left\{\pm\left(\frac{X^2-a^2}{Y^2-a^2}\right)^{\frac{1}{2}}\right\}$. These tangents are real if the given point is within the square, or in the space between two adjacent sides produced. If it lies outside the square, but between two opposite sides produced, XY is a conjugate point. If XY lies upon a side of the square the double point becomes a cusp, the tangent at which is perpendicular to the side.

If X, Y lies upon a diagonal, the locus breaks up into this diagonal, and a circular cubic having its asymptote parallel to the other diagonal.

The equation to the cubic referred to the diagonals is

$$y^2(2x-p) = (x-p)(2x^2-px+2a^2).$$

The three accompanying figures show the shape of the quartic when the point is (1) inside the square $X/1 = Y/3 = 9/4$; (2) between the produced portions of two adjacent sides $X/3 = Y/5 = 9/2$; (3) between two opposite sides produced $X/3 = Y/1 = 9/2$.

8381. (By John Griffiths, M.A.)—If n be a number > 1 and s a number from 1 to $n-1$, show that $1 + 2\Sigma(-)^s \cos^2(s\pi/2n) = 0$.

[The relation in Question follows from the elliptic function formula $y = \sin(L + A + B + C + \ldots)$. See *Proceedings of the London Mathematical Society*, Vol. XVI., No. 238.]

Solution by W. J. Curran Sharp, M.A.

If $S = 1 - \cos(\pi/n) + \cos(2\pi/n) - \ldots + (-1)^{n-1}\cos^{(n-1)}(\pi/n)$,

$$2S\cos(\pi/n) = \cos(\pi/n) - \cos(2\pi/n) + \ldots + (-1)^{n-1}\cos\pi,$$
$$+\cos(\pi/n) - 1 + \cos(\pi/n) \ldots + (-1)^{n-1}\cos^{(n-2)}(\pi/n)$$
$$= 1 + \cos(\pi/n) - 2S - (-1)^{n-1}\{1 + \cos(\pi/n)\},$$

and $S = 0$ if n be odd, and $S = 1$ if n be even.

Now, if n be odd, $1 + 2\Sigma(-)^s \cos^2(s\pi/n) = S = 0$;

if n be even, $1 + 2\Sigma(-)^s \cos^2(s\pi/n) = S - 1 = 0$, therefore, &c.

8454. (By B. Hanumanta Rau, B.A.)—A balloon rises vertically with uniform acceleration; two minutes after starting, when it has attained a height of 1,600 yards, a stone is projected from it horizontally, with a velocity of 32 yards per second; show that the stone will strike the earth at a point 640 yards from the point from which the ascent was made.

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Solution by R. F. Davis, M.A.; Prof. Sarkar, M.A.; *and others.*

Average velocity of balloon = 1,600 yards in two minutes = 40 feet per second, hence upward velocity of balloon at the instant of projection = 80 feet per second. If t seconds be the time of flight,

$$80t - 16t^2 = -4800, \text{ therefore } t = 20 \text{ seconds},$$

whence the stated result follows.

Bibliographie des Questions 8396 et 8516. By Capt. Brocard.

Les propriétés indiquées dans les énoncés 8396 et 8516 ne sont pas toutes nouvelles; on en trouvera la plupart exposées et démontrées dans divers articles déjà anciens.

Le théorème qui fait l'objet de la question 8396 a été proposé pour la première fois à ma connaissance par M. G. Darboux (*Nouvelles Annales de Mathématiques*, 2e Série, T. v. 1886, Quest. 752, p. 95). Dans la solution très complète qui en a été donnée au T. v., p. 420—426, on a fait voir que le lieu cherché est une cubique C_3 ayant pour équation

$$(\alpha + \gamma\cos B)(\beta + \alpha\cos C)(\gamma + \beta\cos A) = (\alpha + \beta\cos C)(\gamma + \alpha\cos B)(\beta + \gamma\cos A),$$

ou $$\beta(\alpha^2 - \gamma^2)(\cos A\cos C - \cos B) + \alpha(\gamma^2 - \beta^2)(\cos B\cos C - \cos A) + \gamma(\beta^2 - \alpha^2)(\cos A\cos B - \cos C) = 0,$$

formes qui se prêtent aisément aux vérifications.

Il y est prouvé que cette C_3 a pour centre le centre O du cercle circonscrit au triangle ABC (p. 422) qu'elle passe par les points I, O_a, O_b, O_c, centres du cercle inscrit et des cercles ex-inscrits (p. 423) et que les asymptotes sont les perpendiculaires aux milieux des côtés (p. 424). M. Gerono observe, en note, que cette C_3 est aussi le lieu du point P tel que les perpendiculaires menées aux droites PA, PB, PC par les sommets A, B, C du triangle rencontrent les côtés opposés BC, AC, AB en trois points en ligne droite, et il ajoute que C_3 passe par l'orthocentre H (p. 424).

M. Darboux n'a pas spécifié les points précités de la courbe, mais il n'est pas douteux qu'il ne les eut remarqués.

Ainsi se trouvent établis les théorèmes de l'énoncé 8396.

En 1876 M. Ed. Lucas a proposé, dans la *Nouvelle Correspondance Mathématique* (T. II., p. 94), un autre problème dans lequel on rencontre aussi la même cubique C_3 :

Enoncé 83: on joint les sommets d'un triangle ABC à un point M et l'on prend les intersections A_1, B_1, C_1 des droites AM, BM, CM avec les côtés BC, AC, AB. Trouver quel est le lieu du point M si les perpendiculaires aux côtés en A_1, B_1, C_1 se coupent en un point N. Trouver aussi le lieu du point N.

M. H. Van Aubel, qui en a donné une solution très complète accompagnée de figures (T. IV., 261–272, 355–356; V., 81–87; VI., 56–65), a fait voir plusieurs propriétés énoncées dans la question 8516, notamment les théorèmes 2. (*Loc. cit.* IV., 266), 3. (IV., 264), 4. (IV., 267), et il y a ajouté les intéressantes propriétés que voici:

1. Le point de concours S des tangentes en A, B, C a pour coordonnées $1/a$, $1/b$, $1/c$, a, b, c désignant, pour simplifier, $\cos A - \cos B\cos C$, etc., et il se trouve aussi sur la tangente en H′ (H′ symétrique de H par rapport à O). (IV., 266).

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2. Ce point S est le transformé isogonal (ou par droites symétriques) du point H′ (IV., 266).

3. La courbe C_3 est à elle-même sa propre transformée isogonale, puisque son équation ne change pas lorsqu'on remplace α, β, γ par $1/\alpha$, $1/\beta$, $1/\gamma$.

Nous ne pouvons mieux faire que de renvoyer le lecteur à cet important travail, qui renferme, comme on le voit, les résultats les plus essentiels énoncés dans les questions 8396 et 8516.

La double question 83 de la *N. C. M.* a été reproduite sous le No. 1207 dans les *Nouvelles Annales* (T. XV. 1876, p. 240) et résolue géométriquement par M. E. DEWULF (*loc. cit.* 550–555).

La courbe C_3 est donc une des plus remarquables qui se rencontrent dans la géométrie du triangle, et il n'est pas surprenant que plusieurs géomètres l'aient déjà étudiée à diverses reprises. Il est d'ailleurs utile que les mêmes questions se représentent ainsi à l'attention des géomètres, car chaque fois elles gagnent quelque terrain.

La même question a été traitée, avec d'autres qui s'y rattachent intimement, par M. SCHOUTE, qui a bien voulu nous en communiquer les résultats en 1884.

Signalons enfin la question 735 de M. THOMSON (*N. A. M.* IV., 1865, p. 144 et 469–473).

[Mr. PURSER states that he was quite unaware that this cubic locus had been already so fully discussed. The following points, however, may not be without interest:—

(α) The cubic belongs to the general class of curves of the third degree defined by the property that the line joining any point on the curve to its isogonal passes through a fixed point (Q). (For the cubic in question the point Q will be found to be that denoted H′ by M. BROCARD.)

(β) All curves of this type pass through the in- and ex-centres, and the four tangents at these points are concurrent in Q.

(γ) For the cubic of Quest. 8396 the inflexional tangent at O is the line joining O to the symmedian of the triangle.]

8597. (By Professor IGNACIO BEYENS.)—Dans un triangle sphérique qui a un angle $A = 60°$, on vérifie la relation entre les autres éléments du triangle, étant égal $a + b + c$,

$$\sin B \sin C = 3 \sin (s-b) \sin (s-c) \,/\, \sin^2 a, \quad 2s.$$

Solution by Professor SCOTT, M.A.; BELLE EASTON; *and others.*

$$\sin B \sin C = \frac{4}{\sin c \sin b \sin^2 a} \sin s \sin (s-a) \sin (s-b) \sin (s-c)$$

$$= \frac{\sin (s-b) \sin (s-c)}{\sin^2 a} \times \frac{4 \sin s \sin (s-a)}{\sin b \sin c},$$

$$\cos^2 \tfrac{1}{2}A = \frac{\sin s \sin (s-a)}{\sin b \sin c} = \tfrac{3}{4}; \quad \text{therefore, \&c.}$$

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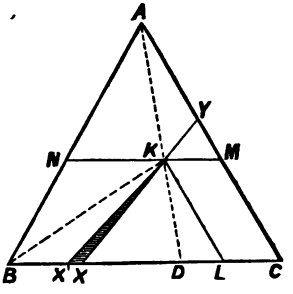
8232. (By Rev. T. C. Simmons, M.A.)—Three points, P, Q, R, are taken at random within a given triangle, also a point K whose triangular coordinates are a, b, c. Prove that (1) the chance that Q and R fall on opposite sides of PK is equal to

$$2a^2 \int_{\frac{ab}{1-a}}^{1-b} \left\{ 2b - \left| 6ab^2 + (1-4ab)\left(x + \frac{b^2}{x}\right) - a\left(x^2 + \frac{b^4}{x^2}\right) \right| \right\} dx,$$

together with the five similar expressions obtained by the interchanging of the letters a, b, c; (2) hence, or otherwise, when K is situated on the perimeter, the required chance becomes $\frac{1}{3}$; and (3) when K is situated at the centroid, it becomes $\frac{31}{81} + \frac{40}{243} \log 2$ or ·4968 nearly.

Solution by the Proposer.

1. It is evident that we need only consider the case of an equilateral triangle. For any other triangle can be transformed into this by orthogonal projection, and since the operation does not affect the mutual ratios of any areas, nor the triangular coordinates of any point, the theorems in question, if proved true for the one case, will follow inferentially for the other.



Take, then, an equilateral triangle ABC, each of whose sides is denoted by unity. Draw NKM parallel to BC and KL parallel to AC. Then, since KL/AC, KM/BC, and KN/BC are the measures of the triangular coordinates of K, we have $KL = a$, $KM = b$, $KN = c$. Draw any line XKY through K, and let $LX = x$. Take any point X′ very near to X, and let $XX' = dx$.

Then, of the three points P, Q, R, the chance that P lies within the elementary area KXX′, and that Q, R lie at the same time on opposite sides of KP, is represented by

$$\frac{2dx \,.\, KL}{BC \,.\, AC} \times \frac{CX \,.\, CY}{CA \,.\, CB} \times \left(1 - \frac{CX \,.\, CY}{CA \,.\, CB}\right)$$

$$= 2dx \,.\, a \times (LX + CL)\left(CM + \frac{KL \,.\, KM}{LX}\right) \times (\ldots)$$

$$= 2dx \,.\, a \times (x+b)\left(a + \frac{ab}{x}\right) \times (\ldots) = 2dx \,.\, a^2 \,.\, \frac{(x+b)^2}{x}\left\{1 - \frac{a(x+b)^2}{x}\right\}.$$

Now $LB = 1-b$, $LD = LB - BD = 1 - b - \frac{c}{1-a} = \frac{ab}{1-a}$.

Hence the chance that P lies within the triangle KBD, and Q, R on opposite sides of KP, is represented by the value of

$$2a^2 \int_{\frac{ab}{1-a}}^{1-b} \left\{ \frac{(x+b)^2}{x} - \frac{a(x+b)^4}{x^2} \right\} dx,$$

which is the same as the expression given in the Question. And by taking the corresponding integrals for the five other such triangles as KBD (which will be obtained by the interchanging of the letters a, b, c), and adding, we shall obtain the *whole* chance that Q and R lie on opposite sides of KP.

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2. When K is on the perimeter, one of its triangular coordinates vanishes. Hence we may take $a = 0$, $b + c = 1$. Of the six expressions mentioned, four will be found to vanish, and the sum of the remaining two is $2c^2 \int_0^1 (x - cx^2)\,dx + 2b^2 \int_0^1 (x - bx^2)\,dx = c^2 + b^2 - \frac{2}{3}(c^3 + b^3)$

$= c^2 + b^2 - \frac{2}{3}(c^2 + b^2 - cb) = \frac{1}{3}(c^2 + 2cb + b^2) = \frac{1}{3}$.

3. When K is at the centroid, $a = b = c = \frac{1}{3}$; and the sum of the integrals, which are all equal, is

$$\frac{1}{9}\int_{\frac{1}{4}}^{\frac{3}{4}} \left(\frac{4}{9} + \frac{5x}{9} + \frac{5}{81x} - \frac{x^2}{3} - \frac{1}{243x^2}\right) dx$$

$$= \frac{4}{729}\left(54 + \frac{2}{8}\frac{3}{5} + 30\log 2 - \frac{6}{8}\frac{3}{5} - \frac{8}{5}\right),$$

which reduces to $\frac{3}{8}\frac{1}{1} + \frac{4}{2}\frac{0}{4}\frac{3}{3}\log_e 2 = \cdot 4968$ nearly.

[The Proposer adds that "the Question was originally devised in the hope of obtaining a new method for solving the following interesting Question 4350 (see Vol. xxxv., p. 34), by Colonel Clarke:—'Three random points being taken within a given triangle, what is the chance that they enclose a point (K) of given triangular coordinates a, b, c?' Since the chance, in 8232, that P and R lie on opposite sides of KQ is the same as the chance that Q and R lie on opposite sides of KP, and that they are both, to all appearances, *independent* events, and it being evident that when both concur, K must lie within the triangle formed by P, Q, R, it would seem as if the result in 4350 ought to be the *square* of the result in 8232. The general expressions are so complicated as to afford no means of testing, but Professor Crofton's Quest. 8119 (solved on p. 81 of Vol. xliv.), being a particular case of 4350, will enable us to make an arithmetical comparison. If 4350 and 8232 were connected as above, it is clear that $\frac{2}{7}\frac{2}{7} + \frac{2}{8}\frac{0}{1}\log 2$ ought to be the *square* of $\frac{3}{8}\frac{1}{1} + \frac{4}{2}\frac{0}{4}\frac{3}{3}\log 2$. By calculation we find, however, that the agreement does not extend beyond the second decimal place, the first being ·2452 and the square of the second ·2468; and are driven to the conclusion that, although at first sight so nearly related, the two questions have in reality no connexion whatever."]

8532. (By Emile Vigarié.)—Dans un triangle ABC on donne un côté BC en grandeur et position. Démontrer que, lorsque la somme des deux autres côtés est donnée, (1) La projection de la bissectrice intérieure de l'angle A sur l'un des côtés de cet angle est constante; (2) Le produit des perpendiculaires abaissées des sommets B et C sur la bissectrice extérieure du même angle est constant.

Solution by Rev. T. Galliers, M.A.; Professor Beyens; *and others.*

1. Since a is given and $b + c$ is given, s is given; hence the length of the straight line indicated

$$= \frac{2bc\cos\frac{1}{2}A}{b+c}.\cos\tfrac{1}{2}A = \frac{2bc}{b+c}\cos^2\tfrac{1}{2}A = \frac{2bc}{b+c}\,\frac{s(s-a)}{bc} = \frac{2s(s-a)}{b+c}$$

$$= \text{constant}.$$

2. The product of the specified perpendiculars

$$= b\cos\tfrac{1}{2}A \,.\, c\cos\tfrac{1}{2}A = bc\cos^2\tfrac{1}{2}A = s(s-a) = \text{constant}.$$

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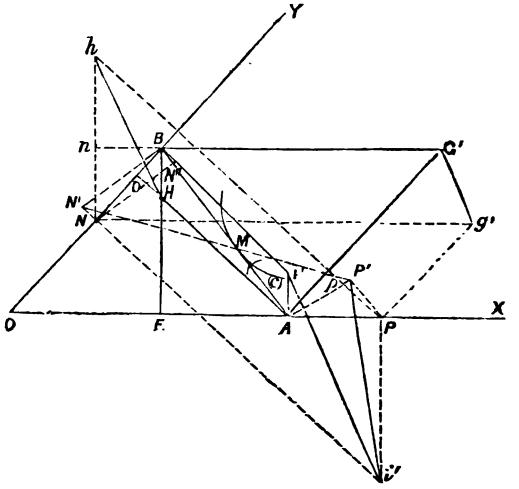
8220. (By Professor NEUBERG.)—On donne une courbe (C) et une angle XOY. Une tangente quelconque à (C) rencontre OX en A, OY en B: trouver (1) la tangente au lieu décrit par l'orthocentre H du triangle OAB; (2) la tangente au lieu décrit par le centre I du cercle OAB; (3) la tangente au lieu du centre de gravité G.

Solutions by the PROPOSER.

1. Soit BN la vitesse de B, que nous décomposons en deux autres BN', BN'' respectivement perpendiculaire à BA et dirigée suivant BA. La 1re BN' est la vitesse de *circulation* de B autour du point de contact M de AB avec (C). Menons N'M qui rencontre en P' une perpendiculaire par A sur AB; AP' est la vitesse de circulation de A autour de M. Menons P'P parallèle à AB, et AP sera la vitesse totale de A.



Nous pouvons aussi décomposer BN en B*n* parallèle à OX, et *n*N perpendiculaire à OX; B*n* sera la vitesse de translation de la hauteur BE. On trouve de même la vitesse de translation A*p* de la hauteur AD.

La vitesse de H dirigée suivant la tangente au lieu décrit par H peut être décomposée (1) parallèlement à OX et perpendiculairement à OX; (2) parallèlement et perpendiculairement à OY. Nous connaissons une composante de chacune de ces décompositions; à savoir B*n* et A*p*. Par suite on peut trouver la vitesse de H; elle est H*h*, si *h* est l'intersection de N*n* et P*p*.

2. Soit I' le point d'intersection de AI', BI' perpendiculaires à OX, OY. Nous connaissons les vitesses de translation de AI' et BI'; c'est AP et BN. Donc les perpendiculaires élevées par P sur OX et par N sur OY se coupent à l'extrémité *i'* de la vitesse de I'.

Les points I et I' ont des vitesses parallèles.

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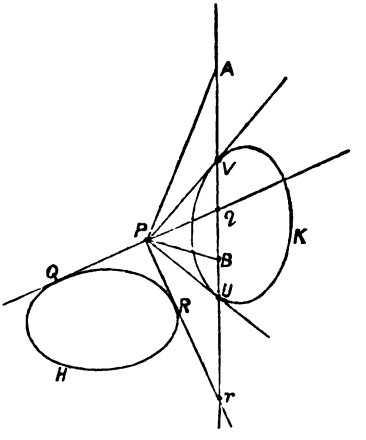
3. Construisons le parallélogramme OAG'B. Les vitesses de translation des droites BG', AG' sont nN, pP. Par suite, le sommet du parallélogramme construit sur NOP donne l'extrémité g' de la vitesse du point G'. Les points G et G' ont des vitesses parallèles.

Autre Solution.—Considérons, par exemple, le lieu décrit par I'. La tangente en I' à cette courbe touche également en I' la courbe que l'on obtiendrait en faisant tourner la droite AB autour du point M au lieu de la faire rouler sur (C). Cette nouvelle courbe est une hyperbole dont on trouve facilement les asymptotes. Car la droite AB, en tournant autour de M, détermine sur OX, OY deux divisions homographiques ; donc les lignes AI', BI' engendrent deux faisceaux parallèles homographiques, et leur point d'intersection décrit une hyperbole passant par les points à l'infini sur AI', BI'. Les tangentes en ces points à l'infini sont les positions des rayons AI', BI' qui correspondent à l'autre rayon transporté à l'infini. Donc, si l'on mène MF et MK parallèles à OX et OZ, FL et KL perpendiculaires à OY et OX, on aura les asymptotes LF et LK. Il reste à mener par I' une droite telle que le segment compris entre LF et LK ait son milieu en I'.

8533. (By Maurice D'Ocagne.)—Si du point P on voit les coniques H et K sous les angles droits, c'est-à-dire, si le point P est à la rencontre des cercles orthoptiques des coniques H et K, et si la polaire du point P par rapport à la conique K coupe en A et en B la polaire réciproque de la conique H par rapport à la conique K, l'angle APB est droit.

Solution by R. F. Davis, M.A.

Let PQ, PR be the perpendicular pair of tangents from P to the conic H ; PU, PV the perpendicular pair to the conic K ; and let the former pair intersect UV in q, r. The points A, B where UV (the polar of P with respect to K) meets the polar reciprocal of H with respect to K will be the poles of PQ, PR with respect to K. Since the anharmonic ratio of a pencil of four concurrent lines is equal to that of the range formed by their four collinear poles with respect to the same conic, therefore



$$P\{UVRA\} = \{UVBq\},$$

or
$$\{UVrA\} = \{UVBq\}$$
$$= \{VUqB\}.$$

Hence the points A, B are a pair in the involution determined by (U, V) and (q, r). But $\angle UPV = qPr$ = a right angle; wherefore APB is also a right angle.

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8444. (By the Editor.)—If four points P, Q, R, S be taken at random, one in each side of a quadrilateral ABCD, find (1) the probability that the area of PQRS will be less than n (area of ABCD), where n is a proper fraction; (2) apply the result to the case of a rectangle, and therefrom (3) show that the probability that PQRS will be less than three-fourths (or, what is the same thing, greater than one-fourth) of ABCD is

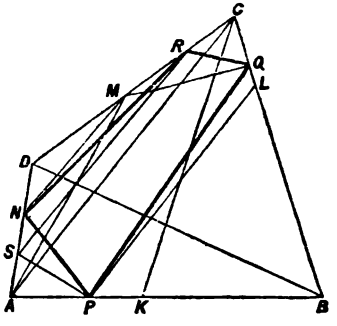
$$\tfrac{1}{2} + \tfrac{5}{4}\log_e 2 = \cdot 991434.$$

Solution by D. Biddle.

Let P, Q, R, S be taken in AB, BC, CD, AD respectively, and let

$x = \mathrm{AP/AB}$, $y = \mathrm{BQ/BC}$,
$z = \mathrm{CR/CD}$, and $u = \mathrm{DS/AD}$;

also let $e = \triangle\mathrm{ABC/ABCD}$,
and $f = \triangle\mathrm{BCD/ABCD}$.



Then PQRS/ABCD

$$= 1-(1-f)(1-u)\,x - e(1-x)\,y - f(1-y)\,z - (1-e)(1-z)\,u$$

$$= 1-\big[(1-u)\,x + (1-z)\,u + e(y-u) + f(z-x) + e(zu-xy) + f(xu-yz)\big] \quad \ldots\ldots\ldots\ldots\ (a).$$

When $x = y = z = u$, the two first terms within the brackets, in (a), are the only ones which do not reduce to zero, and $\mathrm{PQRS/ABCD} = 1-\big[(1-u)\,x + (1-z)\,u\big] = 1-2(1-x)\,x$.

If, under these circumstances, $\mathrm{PQRS/ABCD} = n$, $x = \tfrac{1}{2} \pm \tfrac{1}{2}(2n-1)^{\frac{1}{2}}$; whence it is easy to see that P, Q, R, S cannot be similarly placed on the respective sides of ABCD, unless $\mathrm{PQRS} > \tfrac{1}{2}\mathrm{ABCD}$. The only approach to similarity of position that can be attained when $\mathrm{PQRS} < \tfrac{1}{2}\mathrm{ABCD}$, is represented by $x = 1-y = z = 1-u$, when the sides of PQRS are parallel to the diagonals of ABCD; and in this case, although the two first terms within the brackets, in (a), again remain alone,

$$\mathrm{PQRS/ABCD} = 1-\big[x^2 + (1-x)^2\big],$$

and if now $\mathrm{PQRS/ABCD} = n$, $x = \tfrac{1}{2} \pm \tfrac{1}{2}(1-2n)^{\frac{1}{2}}$. We thus arrive at what may be called the law of symmetrical expansion or contraction of the inscribed quadrilateral. But in order to formulate the probability required, we must find the limits of x, y, z, u, in the equation (a), when $\mathrm{PQRS/ABCD} < n$.

Taking the largest triangle, ABC, formed by two sides and a diagonal of ABCD, draw CK such that $\triangle\mathrm{ACK/ABCD} = n$. (This, of course, is only possible when $n < \triangle\mathrm{ABC/ABCD}$; but we will suppose it for the sake of illustration.) Then, if Q on BC and R on CD coincide with C, and if S on AD also coincide with A, P may reach K without causing the inscribed quadrilateral to exceed n, and if it be placed anywhere nearer A, whilst Q, R, S remain fixed, PQRS/ABCD will be less than n. But $\mathrm{AK/AB} = n/e$, therefore, when $n < \triangle\mathrm{ABC/ABCD}$, the limits of x are 0 and n/e. When $n > \triangle\mathrm{ABC/ABCD}$, the limits of x are 0 and 1. Next, let us suppose P to fall, as in the diagram, between A and K. Draw PL, such

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that BP . BL = BK . BC. Then, since APLC = ACK, it is easy to see that Q may fall anywhere within CL, without causing the inscribed quadrilateral to exceed n (ABCD), provided R, S coincide with C, A, as before. The limits of y, therefore, as represented by BL and BC, are $(e-n) / [e(1-x)]$ and 1. Where $n > e$, these become 0 and 1. Next, let us suppose Q to fall between C and L. Then, in order that AMQP/ABCD may equal n, ΔAMC = QPL + CMQ; in other words,

$$(1-e)\,CM / CD = ey(1-x) - (e-n) + f(1-y)\,CM / CD,$$

whence $$CM / CD = [ey(1-x) - (e-n)] / [(1-e) - f(1-y)],$$

which accordingly is the upper limit of z, 0 being the lower. But it is of importance to observe that, when $f(1-y) > (1-e)$, the upper limit of z is 1. This is the case so soon as Q is sufficiently removed from C to render AQ parallel to CD, and it remains the case during any further removal of Q from C that may be possible.

Finally, let us take R in CM; we then find, by reference to equation (a), that PQRN / ABCD equals n, when

$$DN / AD = [1-n-(1-f)x-e(1-x)y-f(1-y)z] / [(1-e)(1-z)-(1-f)x].$$

Therefore this and 1 are the limits of u, and if S lie in AN the inscribed quadrilateral is less than n (ABCD). But here again it is of importance to observe that, when $(1-f)x > (1-e)(1-z)$, S may be placed anywhere on AD without causing PQRS / ABCD to exceed n, and this is the case when R passes the line drawn parallel to AD through P.

Now, the inscribed quadrilateral might be built up about the diagonal BD as well as about the diagonal AC, and the above-mentioned law of symmetrical development might suffice to convince us that the results obtained in the one way should be doubled in order to arrive at a correct estimate of the whole. But this fact is counter-balanced by another, viz., that the probability as to quadrilaterals less than n is identical with that as to quadrilaterals greater than $(1-n)$. The several differences between the upper and lower limits are the same in either case, and $\frac{1}{2}$ is the meeting point of the two. Moreover, the cases in which $n > \frac{1}{2}$ can readily be transformed into those in which $n < \frac{1}{2}$, and for which the following formulæ are adapted.

Consequently, since the complement of the lower limit of u is the final probability that the inscribed quadrilateral PQRS will fulfil the conditions of the question, when P, Q, R have already fallen within their due limits, —the following integral, if the three variables be taken between the limits hereinbefore indicated, will give the probability required,

$$\iiint \frac{n-e-exy+ey+(e+f-1)z-fyz}{(1-e)(1-z)-(1-f)x}\,dx\,dy\,dz \quad \ldots\ldots\ldots\ldots\ldots\ldots (X).$$

2. In the case of the rectangle, and indeed of any parallelogram (in which $e = f = \frac{1}{2}$), the above integral (X) reduces to

$$\iiint \left(y - \frac{1-2n}{1-x-z}\right) dx\,dy\,dz,$$

and this, for the sake of convenience, can be transformed into

$$\iiint \left(y - \frac{1-2n}{v-z}\right) dv\,dy\,dz,$$

by taking $v = 1-x$, in which case the limits are, for v, $(1-2n)$ and 1;

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for y, $(1-2n)/v$ and 1; for z, 0 and $\left(v-\frac{1-2n}{y}\right)$; and, so far as the rectangle is concerned, there are none of the exceptional limits to which reference has been made.

Continuing the process, the integral becomes

$$\iint\left[y+(1-2n)\left\{\log\frac{1-2n}{y}-\log v\right\}\right]dv\,dy.$$

Let $k=1-2n$, then the integral resolves itself into

$$\iint\left[y+k\left\{\log k-\log y-\log v\right\}\right]dv\,dy$$

$$=\int\left[\tfrac{1}{2}-\frac{k^2}{2v^2}+k\left\{\log k+1+\frac{k}{v}\log\frac{k}{v}-\frac{k}{v}-\log v\right\}\right]dv$$

$$=\tfrac{1}{2}(1-k)(1+3k)+k(1+k)\log k-\tfrac{1}{2}k^2\log^2 k$$

$$=2n(2-3n)+(1-2n)(2-2n)\log(1-2n)-\tfrac{1}{2}(1-2n)^2\log^2(1-2n),$$

which is the probability that the inscribed quadrilateral will be less than n (ABCD), always supposing $n<\frac{1}{4}$.

3. Here, $n=\frac{3}{4}$, and $p=1-(\frac{5}{8}-\frac{3}{4}\log_e 2-\frac{1}{8}\log_e^2 2)=1-\cdot0450821$ $=\cdot9549179$. Hence, under the conditions, the inscribed quadrilateral will be less than $\frac{1}{4}$ABCD, on the average 9 times out of 200.

If x and z be taken from 0 to $\frac{1}{2}$, and y and u from $\frac{1}{2}$ to 1, and if in the former 0, $\frac{1}{4}$, $\frac{1}{2}$, and in the latter $\frac{1}{2}$, $\frac{3}{4}$, 1, be the only positions occupied, the several areas of PQRS (inscribed in a parallelogram) will be as follows, the formula for calculation being derived from (a) above.

$$1-\tfrac{1}{2}(x+y+z+u-xy-yz-xu-uz).$$

$1-\frac{1}{2}(0+\frac{1}{2}+0+\frac{1}{2}-0-0-0-0)=\frac{1}{2}$,
$1-\frac{1}{2}(0+\frac{1}{2}+0+\frac{3}{4}-0-0-0-0)=\frac{3}{8}$,
$1-\frac{1}{2}(0+\frac{1}{2}+0+1-0-0-0-0)=\frac{1}{4}$ (b),

$1-\frac{1}{2}(0+\frac{1}{2}+\frac{1}{4}+\frac{1}{2}-0-\frac{1}{8}-0-\frac{1}{8})=\frac{1}{2}$,
$1-\frac{1}{2}(0+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-0-\frac{1}{4}-0-\frac{1}{4})=\frac{1}{2}$,
$1-\frac{1}{2}(0+\frac{1}{2}+\frac{1}{2}+\frac{3}{4}-0-\frac{1}{8}-0-\frac{3}{16})=\frac{13}{32}$,
$1-\frac{1}{2}(0+\frac{1}{2}+\frac{1}{2}+\frac{3}{4}-0-\frac{1}{4}-0-\frac{3}{8})=\frac{7}{16}$,
$1-\frac{1}{2}(0+\frac{1}{2}+\frac{1}{4}+1-0-\frac{1}{8}-0-\frac{1}{4})=\frac{5}{16}$,
$1-\frac{1}{2}(0+\frac{1}{2}+\frac{1}{2}+1-0-\frac{1}{4}-0-\frac{1}{8})=\frac{3}{8}$,

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$1-\frac{1}{2}(0+\frac{3}{4}+0+\frac{1}{2}-0-0-0-0)=\frac{3}{8}$,
$1-\frac{1}{2}(0+1+0+\frac{1}{2}-0-0-0-0)=\frac{1}{4}$ (b),
$1-\frac{1}{2}(0+\frac{3}{4}+0+\frac{3}{4}-0-0-0-0)=\frac{1}{4}$ (b),
$1-\frac{1}{2}(0+1+0+\frac{3}{4}-0-0-0-0)=\frac{1}{8}$ (a),
$1-\frac{1}{2}(0+\frac{3}{4}+0+1-0-0-0-0)=\frac{1}{8}$ (a),
$1-\frac{1}{2}(0+1+0+1-0-0-0-0)=0$ (a),
$1-\frac{1}{2}(0+\frac{3}{4}+\frac{1}{4}+\frac{1}{2}-0-\frac{3}{16}-0-\frac{1}{8})=\frac{13}{32}$,
$1-\frac{1}{2}(0+1+\frac{1}{4}+\frac{1}{2}-0-\frac{1}{4}-0-\frac{1}{8})=\frac{5}{16}$,

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$1-\frac{1}{2}(x+y+z+u-xy\ -yz\ -xu\ -uz)$.

$1-\frac{1}{2}(0+\frac{3}{4}+\frac{1}{2}+\frac{1}{2}-\ 0\ -\frac{3}{8}\ -\ 0\ -\frac{1}{4})=\frac{7}{16}$,
$1-\frac{1}{2}(0+1+\frac{1}{2}+\frac{1}{2}-\ 0\ -\frac{1}{2}\ -\ 0\ -\frac{1}{4})=\frac{3}{8}$,
$1-\frac{1}{2}(0+\frac{3}{4}+\frac{1}{4}+\frac{3}{4}-\ 0\ -\frac{3}{16}-\ 0\ -\frac{3}{16})=\frac{5}{16}$,
$1-\frac{1}{2}(0+1+\frac{1}{4}+\frac{3}{4}-\ 0\ -\frac{1}{4}\ -\ 0\ -\frac{3}{16})=\frac{7}{32}$ (a),
$1-\frac{1}{2}(0+\frac{3}{4}+\frac{1}{2}+\frac{3}{4}-\ 0\ -\frac{3}{8}\ -\ 0\ -\frac{3}{8})=\frac{3}{8}$,
$1-\frac{1}{2}(0+1+\frac{1}{2}+\frac{3}{4}-\ 0\ -\frac{1}{2}\ -\ 0\ -\frac{3}{8})=\frac{5}{16}$,
$1-\frac{1}{2}(0+\frac{3}{4}+\frac{1}{2}+1-\ 0\ -\frac{3}{16}-\ 0\ -\frac{1}{4})=\frac{7}{32}$ (a),
$1-\frac{1}{2}(0+1+\frac{1}{4}+1-\ 0\ -\frac{1}{4}\ -\ 0\ -\frac{1}{4})=\frac{1}{8}$ (a),
$1-\frac{1}{2}(0+\frac{3}{4}+\frac{1}{2}+1-\ 0\ -\frac{3}{8}\ -\ 0\ -\frac{1}{8})=\frac{5}{16}$,
$1-\frac{1}{2}(0+1+\frac{1}{2}+1-\ 0\ -\frac{1}{2}\ -\ 0\ -\frac{1}{2})=\frac{1}{4}$ (b),

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$1-\frac{1}{2}(\frac{1}{4}+\frac{1}{2}+0+\frac{1}{2}-\frac{1}{8}\ -\ 0\ -\frac{1}{8}-0)=\frac{1}{2}$,
$1-\frac{1}{2}(\frac{1}{2}+\frac{1}{2}+0+\frac{1}{2}-\frac{1}{4}\ -\ 0\ -\frac{1}{4}\ -\ 0)=\frac{1}{2}$,
$1-\frac{1}{2}(\frac{1}{4}+\frac{1}{2}+0+\frac{3}{4}-\frac{1}{8}\ -\ 0\ -\frac{3}{16}-0)=\frac{13}{32}$,
$1-\frac{1}{2}(\frac{1}{2}+\frac{1}{2}+0+\frac{3}{4}-\frac{1}{4}\ -\ 0\ -\frac{3}{8}\ -\ 0)=\frac{7}{16}$,
$1-\frac{1}{2}(\frac{1}{4}+\frac{1}{2}+0+1-\frac{1}{8}\ -\ 0\ -\frac{1}{4}\ -\ 0)=\frac{5}{16}$,
$1-\frac{1}{2}(\frac{1}{2}+\frac{1}{2}+0+1-\frac{1}{4}\ -\ 0\ -\frac{1}{2}\ -\ 0)=\frac{3}{8}$,
$1-\frac{1}{2}(\frac{1}{4}+\frac{1}{2}+\frac{1}{4}+\frac{1}{2}-\frac{1}{8}\ -\frac{1}{8}\ -\frac{1}{8}\ -\frac{1}{8})=\frac{1}{4}$,
$1-\frac{1}{2}(\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{2}-\frac{1}{4}\ -\frac{1}{8}\ -\frac{1}{4}\ -\frac{1}{8})=\frac{1}{4}$,
$1-\frac{1}{2}(\frac{1}{4}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-\frac{1}{8}\ -\frac{1}{4}\ -\frac{1}{8}\ -\frac{1}{4})=\frac{1}{4}$,
$1-\frac{1}{2}(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-\frac{1}{4}\ -\frac{1}{4}\ -\frac{1}{4}\ -\frac{1}{4})=\frac{1}{4}$,
$1-\frac{1}{2}(\frac{1}{4}+\frac{1}{2}+\frac{1}{4}+\frac{3}{4}-\frac{1}{8}\ -\frac{1}{8}-\frac{3}{16}-\frac{3}{16})=\frac{7}{16}$,
$1-\frac{1}{2}(\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{3}{4}-\frac{1}{4}\ -\frac{1}{8}\ -\frac{3}{8}\ -\frac{3}{16})=\frac{15}{32}$,
$1-\frac{1}{2}(\frac{1}{4}+\frac{1}{2}+\frac{1}{2}+\frac{3}{4}-\frac{1}{8}\ -\frac{1}{4}-\frac{3}{16}\ -\frac{3}{8})=\frac{15}{32}$,
$1-\frac{1}{2}(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{3}{4}-\frac{1}{4}\ -\frac{1}{4}\ -\frac{3}{8}\ -\frac{3}{8})=\frac{1}{2}$,
$1-\frac{1}{2}(\frac{1}{4}+\frac{1}{2}+\frac{1}{4}+1-\frac{1}{8}\ -\frac{1}{8}\ -\frac{1}{4}\ -\frac{1}{4})=\frac{3}{8}$,
$1-\frac{1}{2}(\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+1-\frac{1}{4}\ -\frac{1}{8}\ -\frac{1}{8}\ -\frac{1}{4})=\frac{7}{16}$,
$1-\frac{1}{2}(\frac{1}{4}+\frac{1}{2}+\frac{1}{2}+1-\frac{1}{8}\ -\frac{1}{4}\ -\frac{1}{4}\ -\frac{1}{8})=\frac{7}{16}$,
$1-\frac{1}{2}(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+1-\frac{1}{4}\ -\frac{1}{4}\ -\frac{1}{2}\ -\frac{1}{4})=\frac{1}{2}$,
$1-\frac{1}{2}(\frac{1}{4}+\frac{3}{4}+0+\frac{1}{2}-\frac{3}{16}-\ 0\ -\frac{1}{8}-0)=\frac{13}{32}$,
$1-\frac{1}{2}(\frac{1}{2}+\frac{3}{4}+0+\frac{1}{2}-\frac{3}{8}\ -\ 0\ -\frac{1}{4}\ -\ 0)=\frac{7}{16}$,
$1-\frac{1}{2}(\frac{1}{4}+1+0+\frac{1}{2}-\frac{1}{4}\ -\ 0\ -\frac{1}{8}\ -\ 0)=\frac{5}{16}$,
$1-\frac{1}{2}(\frac{1}{2}+1+0+\frac{1}{2}-\frac{1}{2}\ -\ 0\ -\frac{1}{4}\ -\ 0)=\frac{3}{8}$,
$1-\frac{1}{2}(\frac{1}{4}+\frac{3}{4}+0+\frac{3}{4}-\frac{3}{16}\ -\ 0\ -\frac{3}{16}\ -\ 0)=\frac{5}{16}$,
$1-\frac{1}{2}(\frac{1}{2}+\frac{3}{4}+0+\frac{3}{4}-\frac{3}{8}\ -\ 0\ -\frac{3}{8}\ -\ 0)=\frac{3}{8}$,
$1-\frac{1}{2}(\frac{1}{4}+1+0+\frac{3}{4}-\frac{1}{4}\ -\ 0\ -\frac{3}{16}\ -\ 0)=\frac{7}{32}$ (a),
$1-\frac{1}{2}(\frac{1}{2}+1+0+\frac{3}{4}-\frac{1}{2}\ -\ 0\ -\frac{3}{8}\ -\ 0)=\frac{5}{16}$,
$1-\frac{1}{2}(\frac{1}{4}+\frac{3}{4}+0+1-\frac{3}{16}-\ 0\ -\frac{1}{4}\ -\ 0)=\frac{7}{32}$ (a),
$1-\frac{1}{2}(\frac{1}{2}+\frac{3}{4}+0+1-\frac{3}{8}\ -\ 0\ -\frac{1}{2}\ -\ 0)=\frac{5}{16}$,
$1-\frac{1}{2}(\frac{1}{4}+1+0+1-\frac{1}{4}\ -\ 0\ -\frac{1}{4}\ -\ 0)=\frac{1}{8}$ (a),
$1-\frac{1}{2}(\frac{1}{2}+1+0+1-\frac{1}{2}\ -\ 0\ -\frac{1}{2}\ -\ 0)=\frac{1}{4}$ (b),

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$1-\frac{1}{2}(x+y+z+u-xy\ -yz\ -xu\ -uz)$.

$1-\frac{1}{2}(\frac{1}{4}+\frac{3}{4}+\frac{1}{4}+\frac{1}{4}-\frac{3}{16}-\frac{3}{16}-\frac{1}{8}-\frac{1}{8}) = \frac{7}{16}$,

$1-\frac{1}{2}(\frac{1}{4}+\frac{3}{4}+\frac{1}{4}+\frac{1}{4}-\frac{3}{8}-\frac{3}{16}-\frac{1}{4}-\frac{1}{8}) = \frac{11}{32}$,

$1-\frac{1}{2}(\frac{1}{4}+1+\frac{1}{4}+\frac{1}{4}-\frac{1}{4}-\frac{1}{4}-\frac{1}{8}-\frac{1}{8}) = \frac{3}{8}$,

$1-\frac{1}{2}(\frac{1}{4}+1+\frac{1}{4}+\frac{1}{4}-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}-\frac{1}{8}) = \frac{7}{16}$,

$1-\frac{1}{2}(\frac{1}{4}+\frac{3}{4}+\frac{1}{4}+\frac{1}{4}-\frac{3}{16}-\frac{3}{8}-\frac{1}{8}-\frac{1}{4}) = \frac{15}{32}$,

$1-\frac{1}{2}(\frac{1}{4}+\frac{3}{4}+\frac{1}{4}+\frac{1}{4}-\frac{3}{8}-\frac{3}{8}-\frac{1}{4}-\frac{1}{4}) = \frac{1}{2}$,

$1-\frac{1}{2}(\frac{1}{4}+1+\frac{1}{4}+\frac{1}{4}-\frac{1}{4}-\frac{1}{4}-\frac{1}{8}-\frac{1}{4}) = \frac{7}{16}$,

$1-\frac{1}{2}(\frac{1}{4}+1+\frac{1}{4}+\frac{1}{4}-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}) = \frac{1}{2}$,

$1-\frac{1}{2}(\frac{1}{4}+\frac{3}{4}+\frac{1}{4}+\frac{3}{4}-\frac{3}{16}-\frac{3}{16}-\frac{3}{16}-\frac{3}{16}) = \frac{3}{8}$,

$1-\frac{1}{2}(\frac{1}{4}+\frac{3}{4}+\frac{1}{4}+\frac{3}{4}-\frac{3}{8}-\frac{3}{16}-\frac{3}{8}-\frac{3}{16}) = \frac{7}{16}$,

$1-\frac{1}{2}(\frac{1}{4}+1+\frac{1}{4}+\frac{3}{4}-\frac{1}{4}-\frac{1}{4}-\frac{3}{16}-\frac{3}{16}) = \frac{6}{16}$,

$1-\frac{1}{2}(\frac{1}{4}+1+\frac{1}{4}+\frac{3}{4}-\frac{1}{4}-\frac{1}{4}-\frac{3}{8}-\frac{3}{16}) = \frac{11}{32}$,

$1-\frac{1}{2}(\frac{1}{4}+\frac{3}{4}+\frac{1}{4}+\frac{3}{4}-\frac{3}{16}-\frac{3}{8}-\frac{3}{16}-\frac{3}{8}) = \frac{7}{16}$,

$1-\frac{1}{2}(\frac{1}{4}+\frac{3}{4}+\frac{1}{4}+\frac{3}{4}-\frac{3}{8}-\frac{3}{8}-\frac{3}{8}-\frac{3}{8}) = \frac{1}{2}$,

$1-\frac{1}{2}(\frac{1}{4}+1+\frac{1}{4}+\frac{3}{4}-\frac{1}{4}-\frac{1}{4}-\frac{3}{16}-\frac{3}{8}) = \frac{13}{32}$,

$1-\frac{1}{2}(\frac{1}{4}+1+\frac{1}{4}+\frac{3}{4}-\frac{1}{4}-\frac{1}{4}-\frac{3}{8}-\frac{3}{8}) = \frac{1}{2}$,

$1-\frac{1}{2}(\frac{1}{4}+\frac{3}{4}+\frac{1}{4}+1-\frac{3}{16}-\frac{3}{16}-\frac{1}{4}-\frac{1}{4}) = \frac{5}{16}$,

$1-\frac{1}{2}(\frac{1}{4}+\frac{3}{4}+\frac{1}{4}+1-\frac{3}{8}-\frac{3}{16}-\frac{1}{4}-\frac{1}{4}) = \frac{13}{32}$.

$1-\frac{1}{2}(\frac{1}{4}+1+\frac{1}{4}+1-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}) = \frac{1}{4}$ (b),

$1-\frac{1}{2}(\frac{1}{4}+1+\frac{1}{4}+1-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}) = \frac{3}{8}$,

$1-\frac{1}{2}(\frac{1}{4}+\frac{3}{4}+\frac{1}{4}+1-\frac{3}{16}-\frac{3}{8}-\frac{1}{4}-\frac{1}{4}) = \frac{13}{32}$,

$1-\frac{1}{2}(\frac{1}{4}+\frac{3}{4}+\frac{1}{4}+1-\frac{3}{8}-\frac{3}{8}-\frac{1}{4}-\frac{1}{4}) = \frac{1}{2}$,

$1-\frac{1}{2}(\frac{1}{4}+1+\frac{1}{4}+1-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}) = \frac{3}{8}$,

$1-\frac{1}{2}(\frac{1}{4}+1+\frac{1}{4}+1-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}) = \frac{1}{4}$.

The 81 cases here given exhibit PQRS 9 times under $\frac{1}{4}$ (a), and, in addition, 6 times exactly $\frac{1}{4}$ (b); in all, 15 not exceeding that area. But in each side of the quadrilateral only three positions out of five are taken, viz., 0, $\frac{1}{4}$, $\frac{1}{2}$ on the one hand, and $\frac{1}{2}$, $\frac{3}{4}$, 1 on the other hand, out of 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 1. Consequently, to arrive at an approximate estimate, we must adjust the numbers. If all five positions in each side were available, the whole number of cases would be $5^4 = 625$, but out of that number, those giving PQRS $< \frac{1}{4}$ would be twice as many as in the above list, because an equal number of quadrilaterals similar in size and shape would be built up about each diagonal. Therefore we find that, under these rude conditions, 30 quadrilaterals out of 625 would not exceed the prescribed area ($\frac{1}{4}$ABCD), and this proportion is rather more than 9 out of 200.

If we take $n = \frac{7}{32}$, there are in the foregoing list 9 cases (a), giving PQRS $< (\frac{7}{32}$ABCD). This doubled gives 18 out of 625. The above formula gives $p = \frac{3}{5}\frac{9}{1}\frac{1}{2} + \frac{3}{4}\frac{2}{1}\frac{5}{2}\frac{5}{8}\log_e\frac{9}{16} - \frac{8}{5}\frac{1}{1}\frac{1}{2}\log_e{}^2\frac{9}{16} = \cdot02983$, which is sufficiently close to assure us of the correctness of the method adopted.

[This is the first published attempt at a solution of the *general* form of the problem as set forth in part (1) of the Question. Solutions of the *particular* cases in parts (2) and (3) have been published in Vol. I., pp. 93—95, and Vol. XLV., pp. 103, 104.]

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8379. (By B. HANUMANTA RAU, B.A.)—A'B'C' is a triangle inscribed in another triangle ABC and similar to it. Prove that (1) the orthocentre of A'B'C' is fixed and coincides with the circum-centre of ABC; (2) the locus of its circum-centre is a fixed straight line through the nine-point centre of ABC; (3) the locus of the centroid is a straight line through the centroid of ABC; (4) the envelopes of the sides A'B', B'C', C'A' are three parabolas touching the sides of ABC two and two; (5) the circum-circles of the triangles AC'B', BC'A', CA'B', and A'B'C' are all equal; (6) if O_1, O_2, O_3 be the centres of the first three circles, the triangles O_1, O_2, O_3 and A'B'C' are equal in all respects and have the same nine-point circle.

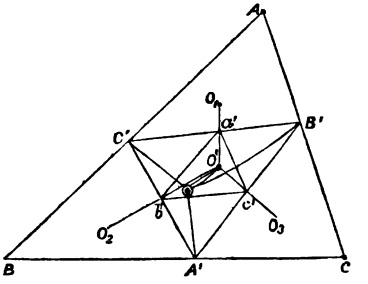
Solution by R. F. DAVIS, M.A.; SARAH MARKS; *and others.*

(1). Let O be the orthocentre of A'B'C'; then

$$\angle B'OC' = \pi - A' = \pi - A,$$

and the circum-circle of the triangle AB'C' passes through O. Similarly circum-circles of the triangles BC'A', CA'B' also pass through O. Also

$$\begin{aligned}\angle BOC &= \angle BOA' + \angle COA' \\ &= \angle A'C'B + \angle A'B'C \\ &= 2\pi - B - C - (\angle BA'C' \\ &\quad + \angle CA'B') = 2A.\end{aligned}$$



Similarly $\angle COA = 2B$, $\angle AOB = 2C$,

and O is the circum-centre of ABC.

(2, 3). Let O' be the circum-centre and R' the circum-radius of A'B'C'. Then $OA' = 2R' \cos A$ or $OA' : O'A'$ is constant and

$$\angle OAO' = B' \sim C' = B \sim C;$$

hence the triangle OA'O' is of given species and has one angular point O fixed while another A' describes a fixed straight line BC. The locus of the third angular point O' is therefore a fixed straight line (Casey); and if N' be the middle point of OO', the locus of N' is also a straight line.

A particular case is when A', B', C' are the middle points of the sides of ABC. The circum-centre of A'B'C' then coincides with the nine-point centre of ABC and the centroid of A'B'C' with the centroid of ABC; and the above loci pass respectively through these points.

(4). The envelope of B'C' is obviously a parabola focus O, and whose tangent at the vertex is the fixed pedal line of O with respect to the triangle AB'C', *i.e.*, the line joining middle points of AB, AC.

(5). The circum-circles of AB'C', BC'A', CA'B' are each equal to circum-circle of A'B'C', as the sides B'C' subtend equal angles A = A' at the circumferences, &c.

(6). O_1, O_2, O_3 are the images of O' with respect to B'C', C'A', A'B' respectively. Hence if a', b', c' be the middle points of these sides, $O_2O_3 = 2b'c' = B'C'$, and they are parallel. Thus O' is the orthocentre of $O_1O_2O_3$ and its nine-point circle is the nine-point circle of A'B'C'.

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8710. (By Professor SYLVESTER, F.R.S.) — For greater brevity, let q, r, s, t denote respectively $\frac{\delta_x^2 y}{2}$, $\frac{\delta_x^3 y}{6}$, $\frac{\delta_x^4 y}{24}$, $\frac{\delta_x^5 y}{120}$. Prove that the general integral of the differential equation

$$(5q^2t - 15qrs + 10r^3)^2 + (2qs - \tfrac{8}{3}r^2)^3 = 0$$

is

$$ax + by + c = \log(dx + ey + f).$$

Solution by ROBERT RUSSELL, M.A.

Denoting $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$, $\frac{d^5y}{dx^5}$ by p, q, r, s, t, respectively, the given equation is $(9q^2t - 45qrs + 40r^3)^2 + 2(3qs - 5r^2)^3 = 0$, and, taking q and p as variables, this becomes

$$\left\{9q^2\frac{d^3q}{dp^3} - 9q\frac{dq}{dp}\frac{d^2q}{dp^2} + 4\left(\frac{dq}{dp}\right)^3\right\}^2 + 2\left\{3q\frac{d^2q}{dp^2} - 2\left(\frac{dq}{dp}\right)^2\right\}^3 = 0.$$

Now, denoting $\frac{\delta y}{dx}$, ..., ..., $\frac{\delta^5 y}{dx^5}$ and q, $\frac{1}{2}\frac{dq}{cp}$, ..., $\frac{1}{6}\frac{d^3q}{cp^3}$ by the letters A, B, C, D respectively, the above equation becomes

$$(A^2D - 3ABC + 2B^3)^2 + 4(AC - B^2)^3 = 0,$$

showing immediately that q is a cubic function of p, having a double factor; thus we have $cq = (p-a)^2(p-b)$, therefore

$$\frac{c\,dp}{(p-a)^2(p-b)} = dx, \quad \frac{cp\,dp}{(p-a)^2(p-b)} = dy,$$

therefore

$$\frac{c\,dp}{(p-a)^2} = d(q - bx + m), \quad -\frac{c}{p-a} = y - bx + m;$$

similarly

$$\frac{c\,dp}{p-a\,.\,p-b} = d(q - ax + n),$$

therefore

$$\frac{c}{a-b}\log\frac{p-a}{p-b} = y - ax + n.$$

Hence, eliminating p, the solution is of the form $ax + by + c = e^{dx+my+n}$.

8764. (By Rev. D. THOMAS, M.A.)—If 1^2, 2^2 ... $(n+1)^2$ be the roots of $f(x) = 0$, prove that $\frac{(n+1)^{2n}-1}{f'(n+1)^2} + \frac{n^{2n}-1}{f'(n^2)} + \ldots \frac{1^{2n}-1}{f'(1^2)} = 1$.

Solution by Professor E. H. CULLEY, M.A.

Write $m = n+1$, the degree of $f(x)$; add to left hand $\frac{1^2-1}{f'(1^2)}$, which $= 0$, and we can then express it as $\Sigma\frac{\alpha^{m-1}}{f'(\alpha)} - \Sigma\frac{1}{f'(\alpha)}$, where α denotes a root. Now, by well-known theorems, $\Sigma\frac{\alpha^{m-r+1}}{f'(\alpha)}$ = sum of homogeneous products of degree r of roots of $\phi(x)$, therefore the first of these expressions $= 1$, since r is zero. Also $\Sigma\frac{\alpha^r}{f'(\alpha)} = 0$ if r be $< m-1$, and here again r is zero. Hence the given result.

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8633. (By Professor HERMITE, LL.D.)—Démontrer la formule

$$D_x \operatorname{sn}(x+a)\operatorname{sn}(x+b) = \frac{\operatorname{sn}^2(x+a)-\operatorname{sn}^2(x+b)}{\operatorname{sn}(a-b)}.$$

Solution by Professor MATHEWS.

$$\begin{aligned} &D_x \operatorname{sn}(x+a)\operatorname{sn}(x+b) \\ &= \operatorname{sn}(x+a)\operatorname{cn}(x+b)\operatorname{dn}(x+b) + \operatorname{sn}(x+b)\operatorname{cn}(x+a)\operatorname{dn}(x+a) \\ &= \frac{\{\operatorname{sn}^2(x+a)-\operatorname{sn}^2(x+b)\}\{1-k^2\operatorname{sn}^2(x+a)\operatorname{sn}^2(x+b)\}}{\operatorname{sn}(x+a)\operatorname{cn}(x+b)\operatorname{dn}(x+b)-\operatorname{sn}(x+b)\operatorname{cn}(x+a)\operatorname{dn}(x+a)} \\ &= \frac{\operatorname{sn}^2(x+a)-\operatorname{sn}^2(x+b)}{\operatorname{sn}\{(x+a)-(x+b)\}} = \frac{\operatorname{sn}^2(x+a)-\operatorname{sn}^2(x+b)}{\operatorname{sn}(a-b)}. \end{aligned}$$

8640. (By Professor BORDAGE.)—Through a point (P) lying in the plane of a given circle (O), a secant is drawn; (I) is the projection of the centre (O) upon the intercepted chord. Draw OI and take upon that line IM = IP. What is the geometric locus of the point M, when the secant varies?

Solution by Professor SCHOUTE; Rev. J. WHITE, M.A.; *and others.*

It is immediately evident that the locus consists of the two circles that may be deduced from the given circle by multiplication of its radii vectores for the pole P with $2^{\frac{1}{2}}e^{\pm\frac{1}{4}\pi}$ (multiplication with $2^{\frac{1}{2}}$ combined to a rotation about P over a right angle in both directions).

8679. (By Professor CATALAN.)—Soit y une fonction *impaire* de x, telle que la quantité $\{(1+y^2)^{\frac{1}{2}}-1\}/y$ reste finie, quand (x) varie entre $(-a)$ et $(+a)$. Cela posé : $\int_{-a}^{+a} \frac{dx}{1+y+(1+y^2)^{\frac{1}{2}}} = a.$

Solution by Professor IGNACIO BEYENS.

Par une simple transformation, nous aurons

$$\begin{aligned} \int_{-a}^{+a} \frac{dx}{1+y+(1+y^2)^{\frac{1}{2}}} &= \int_{-a}^{+a}\left(\frac{1+y}{2y}\right)dx - \int_{-a}^{+a}\frac{(1+y^2)^{\frac{1}{2}}}{2y}\,dx \\ &= \int_{-a}^{+a}\frac{dx}{2y} + \int_{-a}^{+a}\frac{dx}{2} - \int_{-a}^{+a}\left\{\left(1+\frac{y^2}{2}-\frac{y^4}{8}\ldots\right)/(2y)\right\}dx \\ &= a + \int_{-a}^{+a}\left(-\frac{y}{4}+\frac{y^3}{8}\ldots\right)dx. \end{aligned}$$

Mais, (y) étant une fonction impaire de x, les intégrales de $y\,dx$, $y^3\,dx$, $y^5\,dx\ldots$ seront des fonctions paires de (x), c'est-à-dire telles que $f(a)=f(-a)$.

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Par suite intégrant entre les limites $(-a)$ et $(+a)$ la seconde intégrale disparaît et la formule est démontrée.

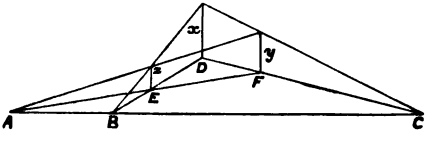
[The PROPOSER remarks that, "Le développement en série, qui n'est peut-être pas *légitime*, est inutile. En effet, A étant l'intégrale cherchée,

$$A = \int_{-a}^{+a} \frac{1+y-\sqrt{1+y^2}}{2y}\,dx = a + \tfrac{1}{2}\int_{-a}^{+a} \frac{1-\sqrt{1+y^2}}{y}\,dx.$$

D'après l'hypothèse $\{1-\sqrt{1+y^2}\}/y$ est une fonction *impaire*, qui reste finie et continue entre $-a$ et $+a$; donc l'intégrale correspondante est nulle, etc.]

8440. (By Professor NILKANTA SARKAR, M.A.)—A, B, C are the three points on a horizontal plane at which the summits of three hills appear to be two and two in the same direction. If x, y, z be the heights of the hills, prove that $\dfrac{x(y-z)}{BC} = \dfrac{y(z-x)}{CA} = \dfrac{z(x-y)}{AB}$.

Solution by C. E. WILLIAMS, M.A.; Prof. IGNACIO BEYENS; *and others.*



We have $BC^2 = BD^2 + CD^2 - 2BD \cdot DC \cdot \cos BDC$;

$$BD = \frac{x}{x-z}ED;\quad CD = \frac{x}{x-y}FD;\quad \cos BDC = \frac{ED^2+FD^2-EF^2}{2ED \cdot FD};$$

$$\therefore BC^2 = \left(\frac{x}{x-z}\right)^2 ED^2 + \left(\frac{x}{x-y}\right)^2 FD^2 - \frac{x^2}{(x-z)(x-y)}(ED^2+FD^2-EF^2);$$

$$\therefore \quad \frac{BC^2}{x^2(y-z)^2} = -\frac{1}{(x-y)(y-z)(z-x)}\left\{\frac{DE^2}{z-x} + \frac{DF^2}{x-y} + \frac{EF^2}{y-z}\right\},$$

which is a symmetrical expression.

8491. (By Rev. T. R. TERRY, M.A.)—If a series $S \equiv a_0 + a_1x + a_2x^2 + \ldots$ is divergent when $x = 1$, find the conditions that $(1-x)S$ may vanish when $x = 1$.

Solution by the PROPOSER.

$$(1-x)S = a_0 + (a_1-a_0)x + \ldots + (a_n-a_{n-1})x^n + \ldots;$$

when $x = 1$, the sum of the first $n+1$ terms of this series is a_n, therefore $(1-x)S = 0$ if $a_n = 0$ when $n = \infty$. This condition is clearly consistent with S being divergent.

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8484. (By C. E. McVicker, M.A.)—If there be given the centre O of a conic that touches the sides of a triangle the mid-points of whose sides are A′, B′, C′, prove that, if OA′, OB′, OC′ intersect B′C′, C′A′, A′B′ in X, Y, Z, then AX, BY, CZ will meet the sides in the points of contact.

Solution by A. Gordon; Rev. T. Galliers, M.A.; *and others.*

Let $L^2\alpha^2 + M^2\beta^2 + N^2\gamma^2 - 2LM\alpha\beta - 2LN\alpha\gamma - 2NM\beta\gamma = 0$ be the conic; and let A′, B′, C′ be the mid-points of sides; then the equation to B′C′ (parallel to BC) is $b\beta + c\gamma - a\alpha = 0$; and if $(\alpha', \beta', \gamma')$ is the centre O,

$$\left.\begin{aligned} M(-M\beta' + L\alpha' + N\gamma') &= \theta b, \\ N(-N\gamma' + L\alpha' + M\beta') &= \theta c \end{aligned}\right\} \ldots\ldots\ldots (1).$$

The line OA′ is $\alpha(c\gamma' - b\beta') + \beta b\alpha' - c\gamma\alpha' = 0$, and to B′C′ is $\beta b + c\gamma - a\alpha = 0$; hence the equation to AX is $c\gamma(-c\gamma' + \beta' b + \alpha' a) - b\beta(a\alpha' - b\beta' + c\gamma') = 0$.

From (1), $2a\alpha' = \dfrac{ab}{LM} + \dfrac{ac}{LN}$, $2b\beta' = \dfrac{ab}{LM} + \dfrac{bc}{MN}$, $2c\gamma' = \dfrac{ac}{LN} + \dfrac{bc}{MN}$;

hence equation to AX is

$$c\gamma\left(\frac{ab}{LM}\right) - b\beta\left(\frac{ac}{LN}\right) = 0, \text{ or } N\gamma - M\beta = 0,$$

which proves the theorem.

8699. (By J. Griffiths, M.A.)—Solve $x^4 + 8x^3 + 6x^2 - 4x - 2 = 0$.

Solutions by (1) R. F. Davis, M.A; (2) D. Edwardes; *and others.*

1. The equation may be written $(x+2)^4 - 18(x+1)^2 = 0$; whence $(x+2)^2 = \pm 3\sqrt{2}(x+1)$, and the solution can be completed.

2. *Otherwise*:—The invariant T vanishes, the Hessian is $-(x^2 + xy + y^2)^2$, and the quartic reduced to its canonical form is $\omega^2(x - \omega y)^4 + \omega(x - \omega^2 y)^4$. Hence the roots are the values of the expression $\omega(1+\alpha)/(1+\alpha\omega^2)$, where ω is an imaginary cube root of unity and α a root of the equation $x^4 + 1 = 0$.

[The Proposer remarks that the Question was meant as an example of the method of solving a biquadratic recently proposed by him. The equation can be reduced to a recurring one by writing $x = m + y$, where m is a root of the cubic

$$(m^4 + 8m^3 + 6m^2 - 4m - 2)(m+2)^2 = (m^3 + 6m^2 + 3m - 1)^2,$$

i.e., $m = 1$. Hence $y^4 + 12y^3 + 36y^2 + 36y + 9 = 0$. Again, write $y = \sqrt{3}.t$, then $t^4 + 4\sqrt{3}(t^3 + t) + 12t^2 + 1 = 0$, or $t = (1 - \sqrt{2} + \sqrt{3} - \sqrt{6})/\sqrt{2}$, so that the value

$$x = 1 + \sqrt{\tfrac{3}{2}}(1 - \sqrt{2} + \sqrt{3} - \sqrt{6}$$
$$= -2 + \frac{3 + \sqrt{3} - \sqrt{6}}{\sqrt{2}} \text{ is a root.}]$$

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8733. (By W. J. C. Sharp, M.A.)—If the $2n$ points in which a conic cuts an n-ic curve be joined two-and-two by n straight lines, prove that the other $n(n-2)$ points in which these lines intersect the n-ic lie upon an $(n-2)$-ic curve.

Solutions by Professors Mathews *and* Nash.

We have two curves of the n^{th} order: viz., the given n-ic curve and the (improper) curve composed of the n straight lines. Through $\frac{1}{2}(n-2)(n+1)$ of the n^2-2n intersections which do not lie on the conic, draw a curve of order $(n-2)$; call this C_{n-2}. Then C_{n-2} and the conic make up an improper curve of the n^{th} order which goes through

$$\tfrac{1}{2}(n-2)(n+1)+2n = \tfrac{1}{2}(n^2+3n-2) = n^2-\tfrac{1}{2}(n-1)(n-2)$$

of the n^2 intersections, and must therefore, by a well-known theorem, pass through the rest; therefore C_{n-2} goes through all the n^2-2n intersections which are not on the conic.

8634. (By Professor Steggall, M.A.)—Show that the general solution in rational quantities of the equation $x^2+xy+y^2=z^2$ is

$$x = p^2-q^2, \quad y = 2pq+q^2, \quad z = p^2+pq+q^2.$$

Solutions by (1) Prof. Neuberg; (2) Prof. Mathews.

1. En écrivant $x(x+y) = (z+y)(z-y)$, on voit qu'on peut poser

$$\frac{x}{z+y} = \frac{z-y}{x+y} = \frac{p}{q},$$

ou
$$qx-py = pz, \quad px+(p+q)y = qz \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

p et q étant deux indéterminées.

De (1) on déduit $\dfrac{x}{z} = \dfrac{p^2-q^2}{p^2+pq+q^2}$, $\dfrac{y}{z} = \dfrac{2pq+q^2}{p^2+pq+q^2}$,

ce qui démontre la question, pourvu que l'on suppose x, y, z premiers entre eux, de même que p et q.

2. *Otherwise*:—Denoting by ω a complex cube root of unity

$$\begin{aligned}(p^2+pq+q^2)^2 &= (p-\omega q)^2(p-\omega^2 q)^2\\ &= \{p^2-2\omega pq-q^2(1+\omega)\}\{p^2-2\omega^2 pq-q^2(1+\omega^2)\}\\ &= \{(p^2-q^2)-\omega(2pq+q^2)\}\{p^2-q^2-\omega^2(2pq+q^2)\}\\ &= x^2+xy+y^2,\end{aligned}$$

if $x = p^2-q^2$, $y = 2pq+q^2$.

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8552. (By Professor NEUBERG.)—On donne un angle XOY et une courbe (C). Une tangente quelconque menée à (C) détermine avec l'angle XOY un triangle OAB. Trouver géométriquement la tangente menée au lieu décrit par l'orthocentre ou par le centre de gravité du triangle OAB.

Solution et extension par Profs. G. DE LONGCHAMPS *and* MATHEWS.

1. Occupons-nous d'abord du lieu décrit par l'orthocentre du triangle AOB. A cet effet, considérons une seconde tangente A'B', infiniment voisine de AB, et désignons par H et H' les orthocentres de ces deux triangles AOB, A'OB'. Ces deux points sont infiniment voisins et il s'agit de fixer la position limite de HH'.

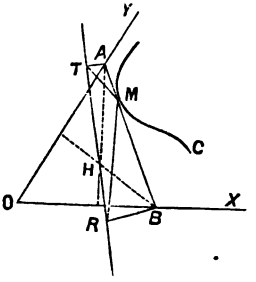

FIG. 1.

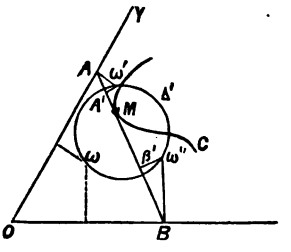

FIG. 2.

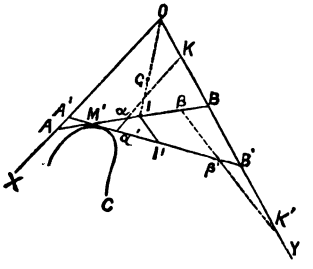

FIG. 3.

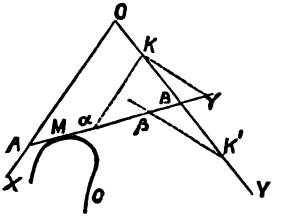

FIG. 4.

Mais si l'on considère le quadrilatère complet formé par les droites OX, OY, AB, A'B', on sait que les orthocentres des quatre triangles déterminés par ces droites, combinées trois à trois, sont quatre points en ligne droite.

De cette simple remarque résulte la construction suivante.

A la droite AB on élève, aux points A et B, des perpendiculaires qui coupent aux points R et T les perpendiculaires abaissées du point de contact M sur les droites OX, OY. Ces points R et T, ainsi construits, représentent, à la limite, les orthocentres des triangles qui, avec AOB et A'OB', complètent les quatre triangles auxquels nous avons fait allusion

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tout à l'heure. La droite RT passe donc par le point H et, en ce point, représente la tangente au lieu décrit par lui.

2. La méthode que nous venons de suivre, pour résoudre la première partie de la question présente, peut être imitée pour le cas que nous allons examiner et dans lequel on propose, dans les mêmes conditions, de trouver la tangente au lieu décrit par le centre du cercle circonscrit au triangle AOB.

Voici la solution de cette question qui complète un peu celle de M. Neuberg.

On sait que, dans un quadrilatère complet, les centres des cercles circonscrits aux quatre triangles qu'il détermine sont situés sur un cercle Δ. D'après cela, si l'on élève comme l'indique la figure des perpendiculaires aux droites OX, OY, aux points B et A; ces droites rencontrent les perpendiculaires élevées à AB aux points A′ et B′, qui partagent en deux parties égales les segments AM, BM, en deux points ω', ω''. Soit ω le centre du cercle circonscrit au triangle AOB. Le cercle $\omega\omega'\omega''$ passe par le point ω, tangentiellement à la courbe, lieu du point ω.

3. La tangente au lieu décrit par le centre de gravité du triangle AOB présente un peu plus de difficultés; mais voici une solution simple basée sur la considération des droites que nous appelons *transversales réciproques*.

Considérons d'abord deux tangentes AB, A′B′ à la courbe proposée C, et soit M′ leur point de concours. Prenons

$$M'\alpha = \beta B = AM', \quad M'\alpha' = \beta' B' = A'M';$$

nous pourrons observer (i.) que $\alpha\alpha'$ est parallèle à AA′, (ii.) que $\alpha\alpha'$ et $\beta\beta'$ étant deux transversales réciproques du triangle M′BB′ coupent BB′ en deux points K, K′, *isotomiques* par rapport à BB′, c'est-à-dire symétriques par rapport au milieu de BB′.

Passons maintenant à la limite et supposons que les deux tangentes considérées viennent se confondre. De la figure (3) nous déduisons la figure (4) dans laquelle nous avons pris, bien entendu, $M\alpha = AM = \beta B$, $BK' = KB$, la droite αK étant d'ailleurs parallèle à OX.

Si nous observons qu'en prenant les points I, I′ milieux des droites AB et A′B′, ces mêmes points sont aussi les milieux des segments $M'\beta$, $M'\beta'$, nous reconnaîtrons donc que II′ est parallèle à $\beta\beta'$, ou que, à la limite, la direction de II′ est parallèle à celle de la droite $K'\beta$ de la figure (4).

Le lieu décrit par le centre de gravité de OAB étant homothétique à celui qui est décrit par I, nous concluons enfin que la direction de la tangente en G est celle de la droite $K'\beta$.

On peut encore observer qu'en prenant $B\gamma = AM$ la droite $K\gamma$ est la direction cherchée. Cette remarque abrège encore un peu la construction qui est, d'après ces diverses considérations, trop évidente pour qu'il soit nécessaire de l'expliciter davantage.

8613. (By Captain H. Brocard.)—D et D′ étant les points dont les distances sont inversement ou directement proportionnelles aux cubes des côtés du triangle, on sait que D est le centre d'homologie du triangle donné ABC et du premier triangle de Brocard A′B′C′, et que D′ est le

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pôle de $\Omega\Omega'$ par rapport au cercle de Brocard. Cela posé, on demande d'établir que le pôle de DD′ relativement au même cercle est sur l'axe d'homologie des triangles ABC, A′B′C′.

Solution by Professor G. DE LONGCHAMPS.

1. Les points D et D′ dont il est ici question ont une signification géométrique que nous signalerons d'abord.

En coordonnées normales, les coordonnées de ces points sont

$$(D) \ldots\ldots a^3x = b^3y = c^3z, \quad (D') \ldots\ldots \frac{x'}{a^3} = \frac{y'}{b^3} = \frac{z'}{c^3};$$

les coordonnées barycentriques de ces points sont donc $1/a^2$, $1/b^2$, $1/c^2$ pour le premier ; et a^4, b^4, c^4 pour l'autre.

D'après cette remarque, le point (D) est, dans le sens que, depuis longtemps, nous attribuons à ce mot, *le réciproque du point de Lemoine ;* quant au point (D′) il est le *point inverse* (*Nouvelles Annales*, 1865, p. 399) de (D), suivant l'expression du Colonel Mathieu, *le conjugué isogonal* (*Journal de Mathématiques spéciales*, 1886, p. 28), comme dit M. Neuberg, ou le *réciproque du second ordre* (*Journal de Mathématiques élémentaires*, 1886, p. 109), comme nous l'appelons aussi.

Enfin, quel que soit le terme employé, les points D et D′ sont associés l'un à l'autre de telle façon que les coordonnées barycentriques (α, β, γ), $(\alpha', \beta', \gamma')$ vérifient les égalités $\frac{\alpha\alpha'}{a^2} = \frac{\beta\beta'}{b^2} = \frac{\gamma\gamma'}{c^2}$. La droite DD′ n'est donc autre chose, c'est le point que nous voulions faire observer d'abord, que la droite qui joint *le réciproque D du point de Lemoine au point inverse de D.*

Il s'agit de démontrer que ce point appartient à l'axe d'homologie du triangle ABC et du premier triangle de Brocard.

2. L'équation de DD′ (en coordonnées barycentriques) est

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ a^4, & b^4, & c^4 \\ \frac{1}{a^2}, & \frac{1}{b^2}, & \frac{1}{c^2} \end{vmatrix} = 0, \quad \text{ou} \quad (1) \ldots\ldots \Sigma a^2 (b^6 - c^6)\alpha = 0.$$

Soient α_0, β_0, γ_0 les coordonnées du point μ pôle de DD′ ; la polaire de μ, par rapport au cercle de Brocard, est représentée par

$$(2) \ldots\ldots \Sigma (2b^2c^2\alpha_0 - b^4\gamma_0 - c^4\beta_0)\alpha = 0.$$

Identifions (1) et (2) et nous avons

$$(A) \ldots\ldots \frac{2b^2c^2\alpha_0 - b^4\gamma_0 - c^4\beta_0}{a^2(b^6 - c^6)} = \frac{2c^2a^2\beta_0 - c^4\alpha_0 - a^4\gamma_0}{b^2(c^6 - a^6)} = \frac{2a^2b^2\gamma_0 - a^4\beta_0 - b^4\alpha_0}{c^2(a^6 - b^6)}.$$

De ces relations on peut déduire une infinité de relations linéaires et homogènes entre α_0, β_0, γ_0, et à chacune de ces combinaisons correspond une droite sur laquelle se trouve le point μ.

Les deux combinaisons qui se présentent le plus naturellement sont les suivantes : $\Sigma \frac{2b^2c^2\alpha - b^4\gamma - c^4\beta}{a^2} = 0$, et $\Sigma a^4 (2b^2c^2\alpha - b^4\gamma - c^4\beta) = 0$.

Ces équations peuvent encore s'écrire

$$\Sigma (2b^4c^4 - a^2c^6 - a^2b^6)\alpha = 0, \quad \text{et} \quad (3) \ldots\ldots \Sigma b^2c^2 (a^4 - b^2c^2)\alpha = 0.$$

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En ajoutant ces deux dernières égalités, on a

$$\Sigma\,(b^4c^4 - a^2c^6 - a^2b^6 + a^4b^2c^2)\,\alpha = 0, \text{ ou } \Sigma\,(b^4 - a^2c^2)(c^4 - a^2b^2)\,\alpha = 0,$$

ou enfin

$$(4) \ldots\ldots \Sigma\,\frac{\alpha}{a^4 - b^2c^2} = 0.$$

Sous cette forme on reconnaît bien l'équation de l'axe d'homologie du triangle ABC et du premier triangle de Brocard.

(3) Remarque.—En résolvant les équations (A), ou préférablement, les équations (3) et (4), qui se présentent sous une forme plus simple et plus symétrique, on trouve, sauf erreur, pour les coordonnées du point μ,

$$(G) \ldots\ldots \frac{\alpha}{a^2\,(a^4 - b^2c^2)^2\,(b^6 - c^6)} = \frac{\beta}{b^2\,(b^4 - c^2a^2)^2\,(c^6 - a^6)} = \frac{\gamma}{c^2\,(c^4 - a^2b^2)^2\,(a^6 - b^6)}.$$

Il est intéressant de rapprocher ces formules, relativement compliquées, de la construction élégante et simple qui conduit au point μ, en partant du point de Lemoine, comme nous l'avons indiqué au commencement.

Du point de Lemoine, on déduit d'abord le réciproque D, puis l'inverse D′ de ce réciproque ; le cercle de Brocard étant déterminé quand on connaît le point de Lemoine, il suffit de prendre le pôle de DD′ par rapport à ce cercle.

Le point D′ étant le pôle de la droite de Brocard, le point μ appartient naturellement à cette dernière droite, et l'on peut dire, en résumé, que μ est à l'intersection de la droite de Brocard avec l'axe d'homologie du triangle donné et du premier triangle de Brocard.

8570. (By W. W. Taylor, M.A.)—Prove that the coefficient of x^{2n} in the expansion of $(1 + x + x^2)^{2n}$ is $\left(\frac{2n!}{n!\,n!}\right)^2 + 2\Sigma\left(\frac{2n!}{r!\,(2n-r)!}\right)^2 \cos\frac{2(n-r)\pi}{3}$, where r can have any value from 0 to $n-1$.

Solution by Rev. T. R. Terry, M.A. ; Isabel Madison ; *and others.*

$$(1 + x + x^2)^{2n} = (1 + xe^{\frac{2}{3}(\pi)i})^{2n} \cdot (1 + xe^{-\frac{2}{3}(\pi)i})^{2n}$$

$$= \left(1 + \Sigma\,\frac{(2n)!}{r!\,(2n-r)!}\,x^r e^{\frac{2}{3}(r\pi)i}\right)\left(1 + \Sigma\,\frac{(2n)!}{r!\,(2n-r)!}\,x^{2n-r} e^{\frac{2}{3}[-(2n-r)\pi]i}\right);$$

therefore coeff. of $x^{2n} = \left\{\frac{(2n)!}{n!\,n!}\right\}^2 + 2\Sigma\left\{\frac{(2n)!}{r!\,(2n-r)!}\right\}^2 \cos\frac{2(n-r)\pi}{3}$.

8581. (By Maurice D'Ocagne.)—Par un point M, pris d'une manière quelconque sur le côté BC du triangle ABC, on mène des parallèles aux côtés AC et AB. Ces droites coupent respectivement aux points B′ et C′ les côtés AB et AC. Démontrer que, si la droite qui joint le sommet

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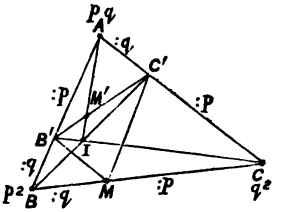
A au point de rencontre I des droites B′C et BC′ coupe la droite B′C′ au point M′, on a $\frac{M'B'}{M'C'} = \frac{MB}{MC}$.

Solution by R. F. Davis, M.A.; G. G. Morrice, M.A.; *and others.*

If MB : MC = $q : p$, the point I must coincide with the centre of gravity (G) of three masses pq, p^2, q^2, placed at the angular points A, B, C respectively. For pq at A and q^2 at C is equivalent to $q(p+q)$ at C′, since AC′ : CC′ = $q : p$; hence G lies on BC′. Similarly G lies on B′C, and therefore coincides with I.



The above system is thus exactly equivalent to masses $-pq$ at A, $p(p+q)$ at B′, and $q(p+q)$ at C′.

Hence M′B′ : M′C′ = $q(p+q) = p(p+q) = q : p$ = MB : MC.

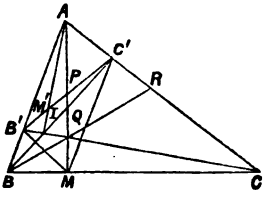
[We may note also that

AI : M′I = $(p+q)^2 : pq$, BI : C′I = $q(p+q) : p^2$, CI : BI′ = $p(p+q) : q^2$,

whence BI . CI : B′I . C′I = AI : M′I,

and BI . B′I : CI . C′I = $MB^3 : MC^3$.

If AM meets B′C′ in P, B′C in Q, the straight line BQ produced bisects AC (by a well-known theorem). Now



$$\{B'M'PC'\} = \{B'IQC\} = \{AC'RC\},$$

i.e., $$\frac{M'B'}{M'C'} = \frac{C'A}{C'C} = \frac{MB}{MC}\Big].$$

8594. (By Professor Clarke, M.A.) —A particle projected from the lowest point of a vertical circular hoop leaves the arc at a point P, and after moving freely in a parabola meets the circle again in P′, the other end of the diameter through P: prove that PP′ must make an angle $\frac{1}{4}\pi$ with the horizon, and that the space due to the velocity of projection is $(1 + \frac{1}{2}\sqrt{2}) \times$ radius of the circle.

Solution by Rev. T. R. Terry, M.A.; Prof. Scott, M.A.; *and others.*

If V = velocity at P, t = time of free motion, θ inclination of PP′ to horizon, then $V^2 = ag \sin\theta$, $2a\cos\theta = Vt\sin\theta$, $2a\sin\theta = \frac{1}{2}gt^2 - Vt\cos\theta$, whence $2a\sin\theta = 2a\cos^4\theta\,\mathrm{cosec}^3\theta$, therefore $\theta = \frac{1}{4}\pi$.

Also velocity at P is due to $\frac{a}{2\sqrt{2}}$, therefore velocity at lowest point is due to $\left(1 + \frac{3\sqrt{2}}{4}\right)a$.

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8615. (By D. Edwardes.)—The space between two coaxal cylinders is filled with homogeneous liquid, and the outer is surrounded by liquid extending to infinity, the whole being bounded by planes perpendicular to the axis. If the inner be suddenly moved perpendicular to its axis, prove that the outer will begin to move with

$$2b^2c^2\rho \,/\, \left[\rho\,(a^2b^2 - a^2c^2 + b^4 + b^2c^2) + \sigma\,(a^2 - b^2)(b^2 - c^2)\right]$$

times the velocity of the inner, where a, b are the external and internal radii of the outer cylinder, σ its density, c the radius of the inner cylinder, and ρ the density of the liquid.

Solution by Professors Sircom *and* Nash.

Let VV_1 be the velocities of A, B the inner and outer cylinders, the distance between the planes unity.

The effect of the inertia of the outer liquid is to increase the mass of B by $\pi\rho a^2$. For the inner liquid the momentum potential due to the motion of B is $\dfrac{V_1 b^2\rho}{b^2 - c^2}\left(\dfrac{b^2}{r} + r\right)\cos\theta$ (see Stokes' *Papers*, Vol. I., p. 36), and the effect is to increase the mass of B by $\pi\rho\,\dfrac{b^2\,(b^2 + c^2)}{b^2 - c^2}\,V_1$.

The momentum of B is therefore

$$\pi\left\{\rho\,\frac{b^2\,(b^2 + c^2)}{b^2 - c^2} + \rho a^2 + \sigma\,(a^2 - b^2)\right\}V_1.$$

The momentum potential due to A being $-\dfrac{Vc^2\rho}{b^2 - c^2}\left(\dfrac{b^2}{r} + r\right)\cos\theta$, the whole resultant pressure parallel to V on the inner surface of B is

$$\pi\rho\,\frac{2b^2c^2}{b^2 - c^2}\,V, \text{ whence } \frac{2b^2c^2\rho}{b^2 - c^2}\,V = \left\{\rho\,\frac{b^2\,(b^2 + c^2)}{b^2 - c^2} + \rho a^2 + \sigma\,(a^2 - b^2)\right\}V_1,$$

which gives the required result.

8386. (By R. Knowles, B.A.)—Tangents TP, TQ meet an ellipse in P and Q; tangents at R, R′, the ends of a diameter, meet TP, TQ in M, N, M′, N′ respectively; if O, O′ be the mid-points of MN, M′N′, prove that OO′, PQ, RR′ are concurrent.

Solution by R. F. Davis, M.A.; Professor Ignacio Beyens; *and others.*

Let PQ, RR′ intersect in U, and draw TZ parallel to MN, M′N′. Then TZ is obviously the polar of U and T $\{$PUQZ$\}$ is harmonic, and TU passes through both O and O′.

8619. (By T. A. E. Sanderson.)—Any point O (h, k) is taken in the plane of a parabola $(y^2 = 4ax)$; and through this point three normals OP,

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OQ, OR are drawn to the curve. If (α, β) be the centre of the circumscribed circle of the triangle PQR, prove that $2\alpha = h + 2a$.

Solution by Rev. T. R. Terry, M.A.; Professor Scott, M.A.; *and others.*

If the coordinates of the three points be $(at_1^2, 2at_1)$, &c., we know (Casey's *Conics*, p. 33), that

$$2\alpha = a(t_1^2 + t_2^2 + t_3^2 + t_1t_2 + t_2t_3 + t_3t_1 + 4).$$

But t_1, t_2, t_3 are the roots of the equation $k + hp - 2ap - ap^3 = 0$, therefore

$$\Sigma t_1 = 0, \quad \Sigma' t_1t_2 = (2a - h) / a, \quad \text{therefore} \quad 2\alpha = 4a - (2a - h) = 2a + h.$$

Geometrical Note. *By* R. Tucker, M.A.

In a Note entitled "The Symmedian-point Axis of an associated system of Triangles," published in the *Quarterly Journal of Mathematics* (Vol. xx., No. 78), (see also Mathematics from *Educational Times*, Vol. xliii., p. 25, 1885), I give the following construction:—

AD, BE, CF are the perpendiculars of the triangle ABC, meeting in the orthocentre P. Take on the sides respectively $B\alpha = CD$, $C\beta = AE$, $B\gamma = AF$; then perpendiculars to the sides BC, CA, AD through α, β, γ cointersect in a point π', the orthocentre of the first "associated" triangle A'B'C' (*i.e.*, with sides through A, B, C parallel to sides of ABC), which lies on the axis containing the ortho-centre, nine-point centre, centroid, and circumcentre.

The object of the present Note is to show that if, with π' as centre, and radius equal the diameter of the polar circle of ABC, a circle be described, then the tangents to it from A, B, C are respectively equal to a, b, c. Take for axes BC and a perpendicular (dO) to it through the circumcentre, and let the circle be given by the equation

$$x^2 + y^2 + 2Gx + 2Fy + H = 0;$$

then, by data, we have

$$c^2 = \tfrac{1}{4}a^2 + Ga + C, \quad b^2 = \tfrac{1}{4}a^2 - Ga + C, \quad a^2 = 2C + 2G^2 + 2Fc \sin B \ldots (1, 2, 3),$$

therefore

$$2Gd = c^2 - b^2, \quad i.e., \quad G = dD \ldots\ldots (4),$$

$$2C = c^2 + b^2 - \tfrac{1}{2}a^2, \quad i.e., \quad C = (\text{median through A})^2,$$

$$-F = 2R(\cos A - \cos B \cos C) \ldots\ldots (5),$$

and

$$\rho^2 = -16R^2 \cos A \cos B \cos C.$$

Equations (4), (5) (see references above) determine the point π', therefore, &c. Other circles may be determined by writing in (1), (2), (3), λc, λb, λa for c, b, a on the sinister side.

Professor Neuberg informs me that some properties of my "Associated" triangles are given in the *Journal de M. de Longchamps* (1886). This gentleman (de Longchamps) calls A'B'C' the "anti-complementary" triangle of ABC, and shows that the circle I have got above is the polar circle of A'B'C', therefore (by a well-known result) it cuts orthogonally the circles on B'C', &c. as diameters, and therefore tangent from $A = A'B' = BC$.

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8688. (By J. J. WALKER, F.R.S.)—If ρ is the radius of curvature at any point of the orbit; r', p' the radius vector from the pole to, and the perpendicular on the tangent at, the corresponding point of the hodograph, show that $f \propto r'^3/\rho p'$.

Solutions by (1) R. F. DAVIS, M.A.; (2) *the* PROPOSER.

1. If χ be the angle which the direction of the acceleration makes at any instant with the direction of motion, $f \sin\chi$ = normal acceleration $= v^2/\rho$. But $\sin\chi = p'/r'$, and $v \propto r'$, therefore, &c.

2. It may be of interest to prove this expression for the acceleration in such a manner as to show what its normal and tangential components are, not to deduce it from them: thus, if P, P′ are two consecutive points in the orbit, described in the infinitesimal time T; Q, Q′ the corresponding points on the hodograph,

$$fr : v = QQ' : r' = QQ'.p' : r'p' = r'PP' : \rho p'$$

(since the angle which QQ′ subtends at the pole is equal to the angle between the normals at P, P′).

Hence $f : v = r'v : \rho p'$ or $f \propto r'^3/\rho p'$. If ϕ is the angle whose sine is p'/r', then, at once, the normal and tangential components of f are v^2/ρ and $v^2 \cot\phi/\rho$.

8487. (By J. O'BYRNE CROKE, M.A.)—If two plane sections of a right cone, S_θ, S_ϕ, pass through the same point of the axis, and be inclined to it respectively at angles θ and ϕ, prove that the radius vector ρ_θ drawn from this point as centre in S_ϕ, and making an angle θ with the transverse axis of the section, is equal to ρ_ϕ in S_θ.

Solution by R. F. DAVIS, M.A.; Rev. T. GALLIERS, M.A.; *and others.*

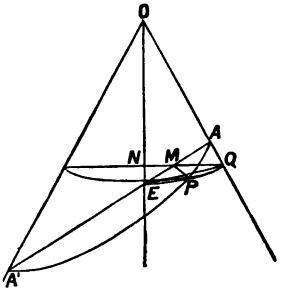
Let AA′ be the transverse axis of S_ϕ passing through the fixed point E on the axis so that $\angle OEA = \phi$; and let EP be the radius vector ρ_θ making with EA the $\angle PEA = \theta$. Draw PM perpendicular on AA′ and QMN on the axis. Then

$$EN = EM \cos\phi,$$

and $EM = EP \cos\theta = EQ \cos\theta$.

Hence $\cos OEQ = \cos\theta . \cos\phi$;

and EQ (or EP), which depends only on the $\angle OEQ$, is the same if θ and ϕ be interchanged.



8438. (By Professor MATHEWS, M.A.)—If P and Q are any two points on the circle circumscribing a triangle ABC, prove that the angles

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of intersection of the pedal lines of P and Q, with regard to the triangle ABC, are equal to the angles which PQ subtends at the circumference of the circle.

Solution by Prof. IGNACIO BEYENS, M.A.; R. F. DAVIS, M.A.; *and others.*

Let O be the orthocentre of the triangle ABC; and HLKI the pedal of P. Then, since B, P, L, H are concyclic,

$\angle PHK = PBA = PGA = HPK$,

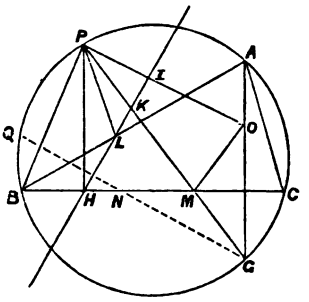
$\therefore \quad KP = KH, \quad \therefore \quad KM = KH$,

$\therefore \angle KHM = KMH = GMC = OMC$;

hence OM is parallel to the pedal of P. Similarly ON is parallel to the pedal of Q; hence the angle contained by the pedals of P and Q

$= \angle MON = MGN$,

i.e., equal to the angle subtended at the circumference by PQ.



[If PP′, QQ′ be chords perpendicular to BC, the pedal lines of P, Q with regard to the triangle ABC are parallel to AP′, AQ′ respectively (see Vol. XXIX., pp. 80–81); hence the angle between these pedal lines is the angle subtended by P′Q′ (or PQ) at the circumference.]

8358. (By the EDITOR.)—If the respective distances of the in-centre from the vertices of a triangle be x, y, z, prove that

$$a^4x^4 + b^4y^4 + c^4z^4 + (a+b+c)^2\,x^2y^2z^2 = 2\,(b^2c^2y^2z^2 + c^2a^2z^2x^2 + a^2b^2x^2y^2).$$

Solution by R. F. DAVIS, M.A.; Prof. M. NATH RAY, M.A.; *and others.*

Here $x \sin \frac{1}{2}A = y \sin \frac{1}{2}B = z \sin \frac{1}{2}C = r$; whence $s^2x^2y^2z^2 = r^2a^2b^2c^2$. Let DEF be the three points of contact of the in-circle; then the sides of DEF are $2ar^2/yz$, $2br^2/zx$, $2cr^2/xy$, and its circum-radius is r. Hence $r \cdot 4\Delta DEF$ = product of sides $= 8abcr^6/x^2y^2z^2$, or $\Delta DEF = 2abcr^5/x^2y^2z^2$. Expressing $\Delta^2(DEF)$ in terms of the sides, the relation of the question can be deduced.

8251. (By Professor MATZ, M.A.)—A candidate is examined in three papers, to each of which m marks are assigned as a maximum; show that, if his total on the three papers is $2m$, there are $\frac{1}{2}(m+1)(m+2)$ ways in which this may occur.

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Solution by W. J. CURRAN SHARP, M.A.

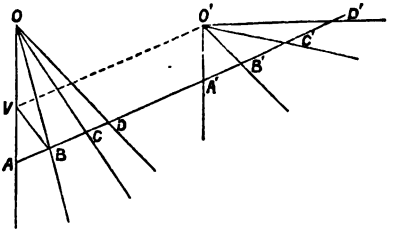
The number of ways is equal to the number of solutions of the equation $\xi+\eta+\zeta=2m$, when ξ, η, ζ are integers not less than zero or greater than m; now this is the coefficient of x^m in $(1+x+x^2\ldots\ldots+x^m)^3=\left(\frac{1-x^{m+1}}{1-x}\right)^3$

$$=(1-3x^{m+1}+3x^{2m+2}-x^{3m+3})\left(1+3x+\ldots\ldots+\frac{m(m+1)}{1.2}x^{m-1}+\ldots\right.$$
$$\left.\ldots\ldots+\frac{(2m+1)(2m+2)}{1.2}x^{2m}+\&c.\right),$$

which is $$\frac{(2m+1)(2m+2)}{1.2}-\frac{3m(m+1)}{1.2}=\frac{m(m+1)}{2}.$$

7392. (By Rev. T. C. SIMMONS, M.A.)—Having given two homographic pencils in a plane, show how to draw a transversal whose segments intercepted between homologous rays shall be equal, each to each.

Solution by the PROPOSER.



Let $O\{ABCD\ldots\ldots\}$,
$O'\{A'B'C'D'\ldots\ldots\}$,
be the pencils; draw one set of homologous parallel rays, OA, O'A' and let O'V', one of the other homologous parallel rays, meet OA in V. Draw VB parallel to O'B' meeting OB in B; through B draw
ABCD ..... A'B'C'D'.....
parallel to OV. Then, this line being parallel to O'V has its intercepted portions AB, BC ..... in a fixed ratio to the portions A'B', B'C' ....., *i.e.*,

$$\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}=\ldots\ldots$$

But AB = A'B'; therefore BC = B'C', CD = C'D' ..... &c.,
and the intercepted portions are *all* equal, each to each.

It is moreover evident that VC, VD ..... will be parallel to O'C', O'D'....., and hence, two homographic pencils being given, it is always possible to move one parallel to itself into such a position that it may have a common transversal with the other. There are obviously *two* solutions.

8526. (By Rev. J. J. MILNE, M.A. Connected with Quest. 8400).—In the base BC of a triangle ABC determine geometrically the position of

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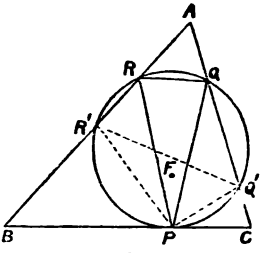
the point P such that, if PQ, PR, containing a given angle, meet AC, AB in Q, R, the area PQAR may be the greatest maximum.

Solution by the PROPOSER.



By Quest. 8400 the triangle PQR is isosceles, and therefore of constant species. Now, in general, if a triangle PQR of constant species be inscribed in another triangle ABC, and the circle round PQR cut the sides of ABC in P', Q', R', the triangle P'Q'R' will be of constant species and will have a permanent centre of similitude F'. (See Proc. of Lond. Math. Soc., Vol. xv., p. 122, and Vol. xvi., p. 184.)

In the case under consideration P' coincides with P, and F' is the intersection of segments of circles on AB, AC containing angles equal to the supplement of Q or R. Also PQ = PR; therefore the angle PQ'Q = PR'R; therefore PQ'C = PR'B. And since F' is also the intersection of circles round BP'R', CP'Q', AQ'R', therefore PF'B = PF'C; therefore the point P is the intersection of BC with the bisector of the angle BF'C.

8464. (By J. BRILL, M.A.)—TP and TQ are tangents to a parabola whose focus is S; H and K are the centres of circles passing through S and touching the parabola at P and Q respectively: and L and M are the centres of the circumscribing circles of the triangles SPT and SQT. O is the mean centre of H, K, L, M for multiples m, m, $3m$, $3m$ respectively. Prove that the circle described with centre O and radius OS touches the parabola at the extremity of the diameter through T.

Solution by Rev. T. R. TERRY, M.A.; Rev. T. GALLIERS, M.A.; *and others.*

If coordinates of P be $[ap^2, 2ap]$, of Q $[aq^2, 2aq]$, of T $[apq, a(p+q)]$; line bisecting SP at right angles is $2x(p^2-1)+4py = a(p^4+4p^2-1)$, line bisecting ST at right angles is

$$2x(pq-1)+2(p+q)y = a(p^2q^2+p^2+q^2+2pq-1);$$

therefore coordinates of L are $\frac{1}{4}a(p^2+1+2pq)$ and $\frac{1}{4}a(2p+q-p^2q)$. So for M. Putting $p=q$, we get coordinates of H are $\frac{1}{4}a(1+3p^2)$ and $\frac{1}{4}ap(3-p^2)$. So for K. And if O be centre of circle which passes through S and touches parabola at point whose coordinates are $\frac{1}{4}a(p+q)^2$ and $a(p+q)$, *i.e.*, at point where diameter through T cuts parabola, coordinates of O are

$$\tfrac{1}{4}a\left[1+\tfrac{3}{4}(p+q)^2\right] \text{ and } \tfrac{1}{4}a(p+q)\left[3-\tfrac{1}{4}(p+q)^2\right].$$

Whence obviously O is mass centre of weights at H, K, L, M proportional to 1, 1, 3, 3.

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8630. (By Prince CAMILLE DE POLIGNAC.)—Prove that the equation $a^n - 2^k = \pm 1$ is impossible in integers unless $n = 1, 2$. [In the latter case $a = 3$.]

Solution by the PROPOSER.

Demonstration.—I premise the following Lemma:—

The index of the highest power of 2 contained in the product $1.2.3 \ldots m$ is $< m$.

Such index is ("Lejeune Dirichlet Zahlentheorie," by Dedekind, 1st ed., p. 29):

$$I = [m/2] + [m/2^2] + [m/2^3] + \ldots,$$

where generally $[m/2^s]$ means the greatest integer contained in $m/2^s$.

Whatever be m, we can place $m = 2^{p_1} + 2^{p_2} + 2^{p_3} + \ldots 2^{p_i}$. The indices are written in the ascending order, and the first can be $= 0$. Then

$$\begin{array}{rccccc} I = & 2^{p_1-1} & + 2^{p_2-1} & + 2^{p_3-1} & + \ldots\ldots & + 2^{p_i-1} \\ & + 2^{p_1-2} & + 2^{p_2-2} & + 2^{p_3-2} & + \ldots\ldots & + 2^{p_i-2} \\ & + 2^{p_1-3} & + 2^{p_2-3} & + 2^{p_3-3} & + \ldots\ldots & + 2^{p_i-3} \\ & \vdots & \vdots & \vdots & & \vdots \\ & 1 & 1 & 1 & \ldots\ldots\ldots & 1 \end{array}$$

$$= 2^{p_1} - 1 + 2^{p_2} - 1 + 2^{p_3} - 1 + \ldots\ldots + 2^{p_i} - 1 = m - i.$$

This shows that the difference between m and I is equal to the number of terms in the above partition of m, and does not depend on the particular values of the indices. The limits are

$i = 1$ corresponding to $m = 2^p$, $I = m - 1$,

$i = p + 1$,, $m = 1 + 2 + 2^2 + \ldots 2^p = 2^{p+1} - 1$, $I = m - 1 - p$.

In all cases $I < m$.

I now take up the given question, working first with the equation $a^n - 2^k = 1$. $a = 2q + 1$ necessarily, and

$$a^n - 1 = n \cdot 2q + \frac{n(n-1)}{1.2} 2^2 q^2 + \ldots \frac{n(n-1)(n-2) \ldots n-m+1}{1.2 \ldots\ldots m} 2^m q^m$$

$$+ \ldots n \cdot 2^{n-1} q^{n-1} + 2^n q^n = 2^k \text{ by hypothesis.}$$

Hence q must be a divisor of 2^k, *i.e.*, a power of 2. Say $q = 2^{h-1}$, we have, dividing by 2^h,

$$2^{k-h} = n + \frac{n(n-1)}{1.2} 2^h + \frac{n(n-1)(n-2)}{1.2.3} 2^{2h} + \ldots$$

$$\ldots + \frac{n(n-1)(n-2) \ldots (n-m+1)}{1.2.3 \ldots m} 2^{(m-1)h} + \ldots + n \cdot 2^{(n-2)h} + 2^{(n-1)h}.$$

Barring the case $n = 1$, which leads to an identical equation, we have $k > h$. On the other hand, according to Lemma, the denominator $1.2.3 \ldots m$ in the general term of the expansion contains a power of 2 smaller than 2^m; therefore, after reduction with $2^{(m-1)h}$, the index of 2 will remain greater than $(m-1)h - m$, say at least $(m-1)h - m + 1 = (m-1)(h-1)$,

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which, except in the first term, where $m = 1$, cannot vanish unless $h = 1$, *i.e.*, $a = 3$, a case which will be considered below.

Therefore, if $a > 3$, all the terms of the expansion except the first contain the factor 2. Hence n must be an even number. Place, then, $n = 2^p . U$, where U is odd. Then 2^p divides obviously out of every term of the expansion which contains n as a factor, *i.e.*, of all but the last; and, if $p < (n-1)\,h$, it will divide out of both sides of the expansion, and, U remaining the only odd number, we shall be led to an impossible equation.

The above condition is here $p < (2^p . U - 1)\,h$, where $h = 2$ at least as $h = 1$ leads to the excluded case $a = 3$. But, even if $U = 1$, we have $p < (2^p - 1)\,2$; hence the condition is always satisfied, and we have shown the indeterminate equation $a^n - 2^k = 1$ to be impossible if $a > 3$.

Suppose now $h = 1$, $a = 3$. The highest power of 2 in $1 . 2 \ldots m$ can only be 2^{m-1} when $m = 2^r$ according to Lemma. In this case the general term of the expansion reduces to

$$\frac{n\,(n-1)(n-2) \ldots\ldots (n-2^r+1)}{1 . 3 . 5 \ldots\ldots (2^r-1)}.$$

It still contains the factor 2. The demonstration goes on as before, and can only be at fault if $n = 1, 2$. In these two cases we have, in fact, $3 - 2 = 1$, $3^2 - 2^3 = 1$.

We now turn to the equation $a^n - 2^k = -1$. As before, a is odd. Take here $a = 2q + 1$; then, if n is even,

$$a^n + 1 = 2^k = 2 - n . 2q + \ldots (-1)^m \frac{n\,(n-1) \ldots (n-m+1)}{1 . 2 \ldots m} 2^m q^m + \ldots 2^n q^n,$$

and as 2 divides out the equation is impossible.

If n is odd, $\quad a^n + 1 = 2^k = n . 2q - \dfrac{n\,(n-1)}{1 . 2} 2^2 q^2 + \ldots$

$$+ (-1)^{m-1} \frac{n\,(n-1) \ldots (n-m+1)}{1 . 2 . 3 \ldots m} 2^m q^m + \ldots + 2^n q^n,$$

an equation which only differs by the signs of the alternate terms from the one met with in the first part of the demonstration, and to which the same reasoning will apply. The case $a = 3$ gives here simply $2^2 - 3 = 1$.

8338. (By Satis Chandra Ray, B.A.)—Solve the equations

$$\sin(\theta + \phi) = a + b, \quad a \tan\phi = b \tan\theta.$$

Solution by Rev. J. J. Milne, M.A.; H. London, B.A.; *and others.*

$a \tan\phi = b \tan\theta$, therefore $a \sin\phi \cos\theta = b \cos\phi \sin\theta$,

$$a + b = \sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi,$$

therefore $a\,(a + b) = a \sin\theta \cos\phi + a \cos\theta \sin\phi = (a + b) \sin\theta \cos\phi$,

therefore $\quad a = \sin\theta \cos\phi$, and $b = \cos\theta \sin\phi$,

therefore $\quad a - b = \sin(\theta - \phi)$, and $a + b = \sin(\theta + \phi)$,

therefore $\theta - \phi = n\pi + (-1)^n \sin^{-1}(a - b)$, $\theta + \phi = m\pi + (-1)^m \sin^{-1}(a + b)$.

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8717. (By Professor HERMITE, LL.D.)—Prouver que l'expression

$$e^{i\pi x/2K}\,\frac{H^2[x+\frac{1}{2}Ki']}{\Theta^2(x)}$$

se ramène à celle-ci : $A \sin x + BD_x \operatorname{sn} x$, A et B étant des constantes. Trouver les valeurs de ces constantes.

Solution by Professor MATHEWS, B.A.

Let $e^{i\pi x/2K}\,\frac{H^2(x+\frac{1}{2}iK')}{\Theta^2(x)} = F(x)$, then we have

$$F(x+2K) = e^{i\pi}F(x) = -F(x), \quad F(x+4K) = F(x).$$

Again, by making use of the formulæ

$$\Theta(x+2iK') = -e^{-i\pi(x+iK')/K}\Theta(x), \quad H(x+2iK') = -e^{-i\pi(x+iK')/K}H(x),$$

we have $F(x+2iK') = e^{i\pi(x+2iK')/2K}\,\frac{e^{-2i\pi(x+\frac{3}{2}iK')/K}\,H^2(x+\frac{1}{2}iK')}{e^{-2i\pi(x+iK')/K}\,\Theta^2(x)} = F(x)$

on reduction, and it is easily seen that $F(x+iK')$ is different from $F(x)$. Hence the simple periods of $F(x)$ are $4K$ and $2iK'$; moreover, the values of x which make $F(x)$ infinite are the roots of $\Theta(x) = 0$, each counting twice. Now, $4K$, $2iK'$ are also simple periods of $\operatorname{sn} x$, and the poles of $\operatorname{sn} x$ are the roots of $\Theta(x) = 0$, each counting once; therefore A, B being constants, we obtain $F(x) = A \operatorname{sn} x + BD_x \operatorname{sn} x$.

To determine A, B, put $x = K$, and $x = 0$, successively, thus

$$A = e^{i\pi}\,\frac{H^2(K+\frac{1}{2}iK')}{\Theta^2(K)} = i\,\frac{H^2(K+\frac{1}{2}iK')}{\Theta^2(K)}, \quad B = \frac{H^2(\frac{1}{2}iK')}{\Theta^2(0)}.$$

8689. (By S. ROBERTS, M.A.)—Given two circles C_1, C_2, show that the centres of circles which (1) are orthogonal to C_1 and are bisected by C_2, or which (2) are orthogonal to C_2 and are bisected by C_1, lie on one and the same circle whose centre is at the mid-point between the centres of the given circles, and which is coaxal with them; also (3) the circle orthogonal to three given circles, the circle which bisects them, and the circle through their centres are coaxal.

Solution by the PROPOSER.

Let s, t be the radii of C_1, C_2, and let a circle C whose radius is r fulfil the conditions (1). Then, denoting by ρ_1, ρ_2 the distance between the centres of C and C_1, and that between the centres of C and C_2, we have $\rho_1^2 = r^2 + s^2$, $\rho_2^2 = -r^2 + t^2$, or $\rho_1^2 + \rho_2^2 = s^2 + t^2$. Now, if the centre of C is considered as variable, this represents a circle which is coaxal with C_1, C_2, and has its centre at the middle point between the centres of C_1, C_2. We may evidently interchange s and t without affecting the result. (3) follows.

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1453. (By Professor SYLVESTER, F.R.S.)—Show that a given point is *inside*, *outside*, or *neither* inside nor outside, a given conic, according as the product of the discriminant by the point-function is *positive*, *negative*, or *zero*.

Solution by W. J. CURRAN SHARP, M.A.

If $\lambda\alpha_1+\mu\alpha_2$, $\lambda\beta_1+\mu\beta_2$, &c. be substituted for α, β, &c., in the tangential equation to a conic

$$\Sigma \equiv A\alpha^2+B\beta^2+C\gamma^2+2F\beta\gamma+2G\gamma\alpha+2H\alpha\beta = 0,$$

the resulting equation $\Sigma_1\lambda^2+2\Pi_{1.2}\lambda\mu+\Sigma_2\mu^2 = 0$

[where $\Pi_{1.2} \equiv A\alpha_1\alpha_2+B\beta_1\beta_2+C\gamma_1\gamma_2+F(\beta_1\gamma_2+\beta_2\gamma_1)+G(\gamma_1\alpha_2+\gamma_2\alpha_1)+H(\alpha_1\beta_2+\alpha_2\beta_1)$]

will determine the tangents drawn from the intersection of

$$\alpha_1x+\beta_1y+\gamma_1z = 0 \text{ and } \alpha_2x+\beta_2y+\gamma_2z = 0$$

to the conic, and these will be imaginary, coincident, or real and distinct, according as $\Sigma_1\Sigma_2-\Pi_{1.2}^2$ is positive, zero, or negative.

Now, if $\alpha_1 = y_1$, $\beta_1 = -x_1$, $\gamma_1 = 0$; $\alpha_2 = z_1$, $\beta_2 = 0$, $\gamma_2 = -x_1$, the point of intersection is (x_1, y_1, z_1), and

$$\begin{aligned}\Sigma_1\Sigma_2-\Pi_{1.2} &= (Ay_1^2+Bx_1^2-2Hx_1y_1)(Az_1^2+Cx_1^2-2Gx_1z_1)\\ &\qquad -(Ay_1z_1+Fx_1^2-Gx_1y_1-Hx_1z_1)^2\\ &\equiv x_1^2\{(BC-F^2)x_1^2+(CA-G^2)y_1^2+(AB-H^2)z_1^2\\ &\qquad +2(GH-AF)y_1z_1+2(HF-BG)z_1x_1+2(FG-CH)x_1y_1\}\\ &\equiv \Delta x_1^2\{ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1\} \equiv \Delta S_1x_1^2.\end{aligned}$$

[There are two cases where a point is neither inside nor outside a conic, viz., (1) when the point is on the conic, and (2) when the conic is a pair of straight lines.]

8638. (By Professor IGNACIO BEYENS.)—Diviser (1) un nombre (n) en quatre parties telles que la première ajoutée a un certain nombre, la seconde retranchée le même nombre, la troisième multipliée par ce nombre et la quatrième divisée par ce nombre, donnent le même résultat; trouver (2) une formule pour résoudre ce problème d'une manière générale; et démontrer (3) que le *nombre* de solutions qu'a la question (en nombres entiers et positifs) est égal au *nombre* des diviseurs carrés qu'a le nombre proposé.

Solution by Prof. MATHEWS, M.A.; W. J. GREENSTREET, B.A.; *and others*.

The conditions of the question being

$$x+a = y-a = az = w/a, \quad x+y+z+w = n;$$

we have $x = a(z-1)$, $y = a(z+1)$, $w = a^2z$,

therefore $2az+a^2z+z = n$, $z = n/(a+1)^2$.

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Thus, if p^2 is any square factor of n, the corresponding parts into which n may be divided will be

$$(p-1)\left\{\frac{n}{p^2}-1\right\},\quad (p-1)\left\{\frac{n}{p^2}+1\right\},\quad \frac{(p-1)^2 n}{p^2},\quad \frac{n}{p^2}.$$

[If solutions are to be in integral positive numbers, the number of solutions depends on the number of times $(a+1)^2$ is contained in n, *i.e.*, on the number of square numbers contained in n.]

7305. (By Professor MALET, F.R.S.)—Prove that the equation of the tangent cone from the point (x', y', z') to the quadric

$$S \equiv ax^2+by^2+cz^2+2hxy+2gxz+2fyz+2lx+2my+2nz+d=0,$$

when referred to its axes, becomes $\lambda_1 x^2+\lambda_2 y^2+\lambda_3 z^2=0$, $\lambda_1, \lambda_2, \lambda_3$ being the roots of the equation

$$\begin{vmatrix} a-\lambda, & h, & g, & U_1 \\ h, & b-\lambda, & f, & U_2 \\ g, & f, & c-\lambda, & U_3 \\ U_1, & U_2, & U_3, & S' \end{vmatrix} = 0, \text{ where } \begin{aligned} U_1 &= ax'+hy'+gz', \\ U_2 &= hx'+by'+fz', \\ U_3 &= gx'+fy'+cz'. \end{aligned}$$

Solution by Professor NASH, M.A.

The equation of the tangent cone is $SS' = P^2$, where

$$P = (xU_1+yU_2+zU_3+d').$$

The coefficients of x^2 and $2yz$ in this equation are $aS'-U_1^2$, $fS'-U_2U_3$, and similarly for the others. The values of $\lambda_1, \lambda_2, \lambda_3$ are therefore the roots of the equation

$$\begin{vmatrix} aS'-U_1^2-\lambda, & hS'-U_1U_2, & gS'-U_1U_3 \\ hS'-U_1U_2, & bS'-U_2^2-\lambda, & fS'-U_2U_3 \\ gS'-U_1U_3, & fS'-U_2U_3, & cS'-U_3^2-\lambda \end{vmatrix} = 0,$$

which is identical with

$$\begin{vmatrix} aS'-U_1^2-\lambda, & hS'-U_1U_2, & gS'-U_1U_3, & U_1 \\ hS'-U_1U_2, & bS'-U_2^2-\lambda, & fS'-U_2U_3, & U_2 \\ gS'-U_1U_3, & fS'-U_2U_3, & cS'-U_3^2-\lambda, & U_3 \\ 0, & 0, & 0, & 1 \end{vmatrix} = 0.$$

Adding to the first three columns the last column multiplied by U_1, U_2, U_3 respectively, and then writing $\lambda S'$ for λ, and dividing the first three columns by S', we get

$$\begin{vmatrix} aS'-\lambda, & hS', & gS', & U_1 \\ hS', & bS'-\lambda, & fS', & U_2 \\ gS', & fS', & cS'-\lambda, & U_3 \\ U_1, & U_2, & U_3, & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} a-\lambda, & h, & g, & U_1 \\ h, & b-\lambda, & f, & U_2 \\ g, & f, & c-\lambda, & U_3 \\ U_1, & U_2, & U_3, & S' \end{vmatrix} = 0.$$

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7361. (By W. J. C. Sharp, M.A.)—In a circular cubic, prove that (1) the directrices, or chords of contact, corresponding to concyclic foci, pass through the centre of the focal circle, which is the point of contact of one of the tangents parallel to the real asymptote; also (2) the four centres lie at the intersections of the cubic and a rectangular hyperbola which has an asymptote in common with the cubic.

Solution by Professor Nash, M.A.

A bicircular quartic is the envelope of a circle whose centre is on a given conic, and which cuts a given circle orthogonally. The bicircular quartic becomes a circular cubic if the conic is a parabola. In either case the points of contact of the variable circle with its envelope lie upon a line passing through the centre of the fixed circle. Since this is true in the general case, it is also true when the radius of the variable circle is evanescent, *i.e.*, when the circle reduces to a focus, and therefore the chord of contact of tangents from a focus passes through the centre of the circle upon which the focus lies.

If A, B, C, D be the four foci lying upon a circle, the four centres of inversion are the centre of the circle ABCD and the vertices of the harmonic triangle of the quadrangle ABCD; and, since each vertex of this triangle is the pole of the opposite side, the centre of the circle is the orthocentre of the triangle. A rectangular hyperbola can therefore be described through the four points. This hyperbola is the locus of the centres of the conics through ABCD, and therefore its asymptotes are parallel to the axes of the two parabolas which can be described through these points. But the axes of the parabolas are perpendicular to the asymptotes of the circular cubics having A, B, C, D as foci, therefore the asymptotes of the rectangular hyperbola are perpendicular (or parallel) to the asymptotes of the two circular cubics having the given points as foci.

8641. (By Professor Neuberg.)—Le cercle osculateur en un point M d'une ellipse rencontre l'ellipse au point M_1; le cercle osculateur en M_1 rencontre l'ellipse en M_2; le cercle osculateur en M_2 la rencontre en M_3, etc.; (1) le polygone $MM_1M_2M_3\ldots$ peut-il se fermer? (2) l'enveloppe de la corde MM_2 est la projection orthogonale d'une épicycloïde.

Solutions by (1) Professor Schoute; (2) Professor Mathews, M.A.

1. The orthogonal projection of an angle and its two bisectors, on a plane parallel to one of these bisectors, is still an angle and its two bisectors. This truth, by the ordinary consideration of the ellipse as the projection of the circle, conducts immediately to the relation $\phi_n = 2\pi - 3\phi_{n-1}$ between the eccentric anomalies ϕ_n and ϕ_{n-1} of the two points M_n and M_{n-1}. The closing condition for a polygon of m sides is therefore $\phi_1 = \frac{2k\pi}{(-3)^{m-1}-1}$, where k represents an arbitrary integer.

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When two points P, Q move in a circumference, and the ratio of the velocities of these points is constant, then the envelope of the line PQ is an epicycloid when the angular velocities of the points are of the same direction, and an hypocycloid in the opposite case. In connection with this theorem of ECKARDT's (SCHLÖMILCH's *Zeitschrift für Mathematik und Physik*, tome 15, page 129), the relation $\varphi_n = 2\pi - 3\phi_{n-1}$ proves that the line in the plane of the circle the projection of which on the plane of the ellipse is the line $M_i M_{i+p}$ envelopes an epicycloid for p constant, etc.

2. *Otherwise*:—Let ϕ be the eccentric angle of M, then (SALMON's *Conics*, Art. 244) we may take -3ϕ for that of M_1, 9ϕ for that of M_2..., $(-3)^r\phi$ for that of M_r. The polygon will be closed if $(-3)^r\phi = \phi \pm 2m\pi$, r, m being positive integers.

The equation of MM_2 is $(x/a)\cos 5\phi + (y/b)\sin 5\phi - \cos 4\phi = 0$. The envelope is the orthogonal projection of the envelope of

$$x\cos 5\phi + y\sin 5\phi - a\cos 4\phi = 0,$$

and this latter envelope is an epicycloid.

8516. (By Professor SIRCOM, M.A.)—In the cubic of Quest. 8396, prove that (1) the tangent at the centre (the real point of inflexion) passes through the Symmedian-point of the triangle; (2) the tangents at the vertices meet in a point; (3) if we take the point on the curve at the other extremity of the diameter through the orthocentre, and from it draw lines to the vertices, these will meet the sides of the triangle in points which are the remaining points of section by the curve; (4) the tangents at the excentres meet in pairs on the lines of (3), also each of these tangents meets the tangent at the incentre on one of these lines.

Solution by the PROPOSER.

(1) The equation being

$$(\beta + \gamma\cos A)(\gamma + \alpha\cos B)(\alpha + \beta\cos C) = (\gamma + \beta\cos A)(\alpha + \gamma\cos B)(\beta + \alpha\cos C),$$

the tangent at α_1, β_1, γ_1 is

$$\alpha\{(\beta_1 + \gamma_1\cos A)(\gamma_1 + 2\alpha_1\cos B + \beta_1\cos B\cos C) - (\gamma_1 + \beta_1\cos A)(\beta_1 + 2\alpha_1\cos C + \gamma_1\cos B\cos C)\} + \text{two others} = 0.$$

Substituting the coordinates of the centre, the equation reduces to

$$\alpha\sin(B - C) + \beta\sin(C - A) + \gamma\sin(A - B) = 0,$$

passing through the Symmedian-point.

(2) The tangents at the vertices meet in the point

$$\alpha(\cos A - \cos B\cos C) = \beta(\cos B - \cos A\cos C) = \gamma(\cos C - \cos A\cos B),$$

as is easily seen by substitution.

(3) Putting $\alpha = 0$ in the curve, we have

$$\beta(\cos C - \cos A\cos B) = \gamma(\cos B - \cos A\cos C),$$

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and the three lines whose equations are thus found meet in the point

$$\frac{\alpha}{\cos A - \cos B \cos C} = \frac{\beta}{\cos B - \cos A \cos C} = \frac{\gamma}{\cos C - \cos A \cos B}.$$

Substituting these in the equation to the diameter through $\frac{\alpha}{\cos A}$ = &c., $\alpha \cos A$ = &c., which is $\alpha \frac{\cos^2 B - \cos^2 C}{\cos B \cos C}$ + two others = 0, we obtain

$$\frac{\cos^2 A\,(\cos^2 B - \cos^2 C)}{\cos A \cos B \cos C} - \cos^2 B + \cos^2 C + \&c.,$$

which vanishes identically.

(4) Let $-\alpha = \beta = \gamma$, $\alpha = -\beta = \gamma$ be two of the excentres. The tangents will be

$$\begin{aligned} &-\alpha\,(\cos B - \cos A \cos C - \cos C + \cos A \cos B) \\ &\quad + \beta\,(\cos C - \cos A \cos B + \cos A - \cos B \cos C) \\ &\quad - \gamma\,(\cos A - \cos B \cos C + \cos B - \cos A \cos C) = 0, \end{aligned}$$

and this and the similar equation for $\alpha = -\beta = \gamma$ give, by addition,

$$-\alpha\,(\cos B - \cos A \cos C) + \beta\,(\cos A - \cos B \cos C) = 0,$$

but this is the equation of one of the above-mentioned lines. Similarly, the tangents to $\alpha = \beta = \gamma$ and $-\alpha = \beta = \gamma$ meet on the line

$$\frac{\beta}{\cos B - \cos A \cos C} = \frac{\gamma}{\cos C - \cos A \cos B}.$$

Similar statements hold for diametrically opposite points.

8421. (By P. C. Ward, M.A.)—Show that the result of eliminating x from the equations $a\,(x+1)^n = (-1)^n c$, $ax^n = b$, is $a^{1/n} + b^{1/n} + c^{1/n} = 0$.

Solution by W. J. Barton, M.A.; W. J. Greenstreet, B.A.; *and others.*

We have

$$x + 1 = -(c/a)^{1/n},\ x = (b/a)^{1/n},\ \text{therefore}\ (b/a)^{1/n} + 1 + (c/a)^{1/n} = 0,$$

which is the stated result.

8637. (By Professor Hudson, M.A.)—The tangent and normal to $y/c = \frac{1}{2}(\epsilon^{x/c} + \epsilon^{-x/c})$ cut the axis of x in T, G respectively; find the minimum value of GT.

Solution by Rev. T. R. Terry, M.A.; Prof. Ignacio Beyens; *and others.*

We have $TN = y \cot \psi$, $NG = y \tan \psi$, therefore

$$TG = \frac{y}{\sin \psi \cos \psi} = \frac{c}{\sin \psi \cos^2 \psi}.$$

The maximum value of $\sin \psi \cos^2 \psi$ is $2/3\sqrt{3}$, therefore minimum value of TG is $3\sqrt{3}\,c/2$.

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8482. (By W. J. M'Clelland, M.A.) – If ABC be a spherical triangle, O the orthocentre, prove that (1) the radius ρ of the polar circle is given by the equation $-\tan^2\rho = (\tan a \tan b \tan c \cos A \cos B \cos C) / 2N$; and deduce the analogous expression *in plano* (see Townsend's *Modern Geometry*, Art 168; $2N = \sin B \sin C \sin a = \sin C \sin A \sin b = \sin A \sin B \sin c$); (2) the polar circles of the triangles BOC, COA, and AOB have their poles respectively at A, B, C and intersect orthogonally; and (3) the perpendiculars of the triangle ABC are the radical axes of the circles taken two and two, and the orthocentre O is the radical centre of the three polar circles of BOC, COA, and AOB.

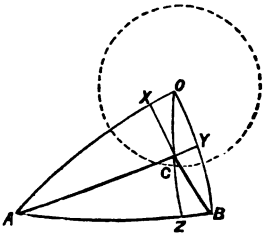
Solution by the Proposer.

1. Let X, Y, Z denote the feet of the perpendiculars, then

$$\frac{\tan CO}{\tan CX} = \sec OCX = \frac{\tan BC}{\tan CZ} \quad \ldots(1),$$

$$\frac{\tan OZ}{\tan CZ} = \frac{\tan OAZ}{\tan CAZ} = \frac{\tan BX}{\sin AX \tan CAZ} \quad \ldots\ldots\ldots\ldots(2),$$

by the formulæ connecting the parts of right-angled triangles.



Multiplying (1) and (2) together, we have

$$\frac{\tan CO \tan OZ}{\tan CX \tan CZ} = \frac{\tan BC \tan BX}{\sin AX \tan CZ \tan CAZ},$$

therefore

$$\tan CO \tan OZ = \frac{\tan a \tan BX \tan CX \cos A}{\sin p \sin A},$$

or

$$-\tan^2\rho = \frac{\tan a \tan b \tan c \cos A \cos B \cos C}{2N}.$$

When the radius of the sphere becomes so great that the powers of the radius higher than the second may be neglected, it is clear that

$$\rho^2 = -d^2 \cos A \cos B \cos C,$$

or the equivalent expression $\frac{1}{2}(a^2+b^2+c^2)-d^2$

(see Townsend's *Modern Geometry*, p. 221).

2. Let ρ_1, ρ_2, ρ_3 denote the radii of the polar circles of the triangles BOC, COA, and AOB, whose poles are at A, B, C.

Then, since $\tan^2\rho_1 = \tan AY \tan AC = \tan AC \tan AB \cos A$,

therefore

$$\tan^2\rho_1 = \tan b \tan c \cos A,$$

$$\tan^2\rho_2 = \tan c \tan a \cos B, \quad \tan^2\rho_3 = \tan a \tan b \cos C,$$

by adding unity to each side of the equations $\sec^2\rho_1 = \frac{\cos a}{\cos b \cos c}$, with similar values for ρ_2 and ρ_3; therefore

$$\cos a \cos\rho_1 = \cos b \cos\rho_2 = \cos c \cos\rho_3 = (\cos a \cos b \cos c)^{\frac{1}{2}}.$$

Hence $\cos a \cos b \cos\rho_1 \cos\rho_2 = \cos a \cos b \cos c$, *i.e.*, $\cos\rho_1 \cos\rho_2 = \cos c$, or c, ρ_1, ρ_2 form a right-angled triangle.

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3. Again, from (2), it follows that

$$\frac{\cos\rho_1}{\cos\rho_2} = \frac{\cos b}{\cos a} = \frac{\cos AZ}{\cos BZ},$$

therefore the perpendiculars AX, BY, CZ of the triangle ABC are the common chords (*i.e.*, radical axes) of the circles of radii ρ_2, ρ_3; ρ_3, ρ_1; ρ_1, ρ_2, and consequently the orthocentre of the triangle is the radical centre of the circles ρ_1, ρ_2, ρ_3.

8602. (By the Editor.)—If three points are taken at random, one on each side of a given triangle, find the average area of the circle drawn through these three points.

Solution by D. Biddle.

Let P, Q, T be the points taken in AB, AC, BC respectively. Let $x = AP/AB$, $y = BT/BC$, $z = CQ/AC$; also let $u = 1-x$, $v = 1-y$, $w = 1-z$, and let $l = 2\cos A$, $m = 2\cos B$, $n = 2\cos C$. Then

$$PQ^2 = c^2x^2 + b^2w^2 - bclxw, \quad QT^2 = a^2v^2 + b^2z^2 - abnvz,$$

$$PT^2 = a^2y^2 + c^2u^2 - acmyu.$$

Now πR^2 = area of circle circumscribing PQT

$$= \pi \cdot \frac{PQ^2 \cdot QT^2 \cdot PT^2}{16\Delta \cdot PQT} = \frac{\pi \cdot PQ^2 \cdot QT^2 \cdot PT^2}{4b^2c^2\sin^2 A\,(1-x-y-z+xy+xz+yz)^2}.$$

By multiplying into each other the above values, we obtain for $PQ^2 \cdot QT^2 \cdot PT^2$, the following 27 terms:—

(1) $+a^2b^2c^2x^2y^2z^2$,	(2) $+a^2b^2c^2u^2v^2w^2$,	(3) $+a^4c^2x^2y^2v^2$,
(4) $+a^4b^2y^2v^2w^2$,	(5) $+a^2b^4y^2z^2w^2$,	(6) $+a^2c^4x^2u^2v^2$,
(7) $+b^2c^4x^2z^2u^2$,	(8) $+b^4c^2z^2u^2w^2$,	(9) $-a^4bclxy^2v^2w$,
(10) $-abc^4nx^2zu^2v$,	(11) $-ab^4cmyz^2uw^2$,	(12) $-a^2b^3clxy^2z^2w$,
(13) $-a^3bc^2nx^2y^2vz$,	(14) $-ab^2c^3mx^2yz^2u$,	(15) $-a^2bc^3lxu^2v^2w$,
(16) $-ab^3c^2nzu^2vw^2$,	(17) $-a^3b^2cm\ uv^2w^2$,	(18) $-a^3b^3ny^2zvw^2$,
(19) $-b^3c^3lxz^2u^2w$,	(20) $-a^3c^3mx^2yuv^2$,	(21) $+a^3b^2clnxy^2zvw$,
(22) $+ab^2c^2lmxyz^2uw$,	(23) $+a^2bc^2mnx^2yzuv$,	(24) $+ab^2c^2lnxzu^2vw$,
(25) $+a^2bc^2lmxyuv^2w$,	(26) $+a^2b^2cmnyzuvw^2$,	(27) $-a^2b^2c^2lmnxyzuvw$.

But it is easy to see that for purposes of integration these terms may at once be reduced to seven, for (1) and (2) are similar, also (3)—(8), (9)—(11), (12)—(17), (18)—(20), (21)—(26). Let the coefficients of the several groups be denoted by e, f, g, h, i, k, p, in which

$$e = 2a^2b^2c^2, \qquad f = a^4c^2 + a^4b^2 + a^2b^4 + a^2c^4 + b^2c^4 + b^4c^2,$$

$$g = a^4bcl + abc^4n + ab^4cm, \quad h = a^2b^2cl + a^3bc^2n + ab^2c^3m + a^2bc^3l + ab^3c^2n + a^3b^2cm,$$

$$i = a^3b^3n + b^3c^3l + a^3c^3m,$$

$$k = a^3b^2cln + ab^3c^2lm + a^2bc^3mn + ab^2c^3ln + a^3bc^2lm + a^2b^3cmn, \quad p = a^2b^2c^2lmn.$$

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Then the seven terms are as follows :—

$$ex^2y^2z^2 + fx^2y^2v^2 - gxy^2v^2w - hxy^2z^2w - iy^2zvw^2 + kxy^2zvw - pxyzuvw,$$

whence we get, by reduction and collocation of similar terms,

$$\begin{aligned}(e+p)\,x^2y^2z^2 &+ (h+k)\,xy^2z^3 - (h+k+3p)\,xy^2z^2 - 2gxyz^4\\ &+ (2g-k)\,xyz^3 + (3p-g+k)\,xyz^2 - pxyz + ix^2y^2\\ &+ 2fx^2y^4 - (2f+3i)\,x^2y^3 + (f+2i)\,x^2y^2 + ixy^3 - ixy^2.\end{aligned}$$

To each of these terms there is the common denominator

$$(1-x-y-z+xy+xz+yz)^2,$$

and also the common factor $\pi/(4b^2c^2\sin^2 A)$, both above mentioned. The limits of the three variables are in each case 0 and 1.

Disregarding for the present the common factor referred to, we find that

$$(e+p)\iiint \frac{x^2y^2z^2\,dx\,dy\,dz}{\Delta^2}$$

$$= (e+p)\iint\left\{\frac{xy}{1-x-y} + 2\left(\frac{xy}{1-x-y}\right)^2 + 2\left[\left(\frac{xy}{1-x-y}\right)^2 + \left(\frac{xy}{1-x-y}\right)^3\right]\right.$$

$$\left.\times\left(\log\frac{x}{1-x} + \log\frac{y}{1-y}\right)\right\}dx\,dy \qquad \ldots\ldots(1).$$

Similarly, (inasmuch as it matters not, for purposes of integration, to what variables the several powers are affixed, and

$$\iiint\frac{xy^2z^3}{\Delta^2} = \iiint\frac{x^3yz^2}{\Delta^2}, \quad \text{also} \quad \iiint\frac{x^2y^3}{\Delta^2} = \iiint\frac{x^3z^2}{\Delta^2}, \text{ \&c., \&c.)}$$

$$(h+k)\iiint\frac{xy^2z^3}{\Delta^2} = (h+k)\iint\left\{\frac{x^2}{1-x-y} + \frac{2x^3y}{(1-x-y)^2}\right.$$

$$\left. + 2\left(\frac{x^3y}{(1-x-y)^2} + \frac{x^4y^2}{(1-x-y)^3}\right)\left(\log\frac{x}{1-x} + \log\frac{y}{1-y}\right)\right\}dx\,dy \ldots(2).$$

$$(h+k+3p)\iiint\frac{xy^2z^2}{\Delta^2} = (h+k+3p)\iint\left\{\frac{y}{1-x-y} + \frac{2xy^2}{(1-x-y)^2}\right.$$

$$\left. + 2\left(\frac{xy^2}{(1-x-y)^2} + \frac{x^2y^3}{(1-x-y)^3}\right)\left(\log\frac{x}{1-x} + \log\frac{y}{1-y}\right)\right\}dx\,dy \ldots(3).$$

$$2g\iiint\frac{xyz^4}{\Delta^2} = 2g\iint\left\{\frac{x^3}{1-x-y} + \frac{x^4y}{(1-x-y)^2}\left(\log\frac{x}{1-x} + \log\frac{y}{1-y}\right)\right\}dx\,dy$$

$$\ldots\ldots(4).$$

$$(2g-k)\iiint\frac{xyz^3}{\Delta^2} = (2g-k)\iint\left\{\frac{x^2}{1-x-y} + \frac{x^3y}{(1-x-y)^2}\left(\log\frac{x}{1-x} + \log\frac{y}{1-y}\right)\right\}$$

$$\times\,dx\,dy \ldots\ldots(5).$$

$$(3p-g+k)\iiint\frac{xyz^2}{\Delta^2} = (3p-g+k)\iint\left\{\frac{1}{1-x-y} + \frac{2xy}{(1-x-y)^2}\right.$$

$$\left. + 2\left(\frac{xy}{(1-x-y)^2} + \frac{x^2y^2}{(1-x-y)^3}\right)\left(\log\frac{x}{1-x} + \log\frac{y}{1-y}\right)\right\}dx\,dy \ldots\ldots(6).$$

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$$p\iiint\frac{xyz}{\Delta^2}=p\iint\left\{\frac{1}{1-x-y}+\frac{xy}{(1-x-y)}\left(\log\frac{x}{1-x}+\log\frac{y}{1-y}\right)\right\}dx\,dy\ldots(7).$$

$$i\iiint\frac{x^3y^3}{\Delta^2}=i\iint\left\{\frac{x^2}{1-x-y}+\frac{9x^3}{2(1-x-y)^2}+\frac{3x^4y}{(1-x-y)^3}\right.$$
$$\left.+3x^3\left(\frac{(1-x-y+xy)^2}{(1-x-y)^4}\right)\left(\log\frac{x}{1-x}+\log\frac{y}{1-y}\right)\right\}dx\,dy\ldots\ldots(8).$$

$$2f\iiint\frac{x^2y^4}{\Delta^2}=2f\iint\left\{\frac{x^3}{y(1-x-y)}+\frac{2x^4}{(1-x-y)^2}\right.$$
$$\left.+2\left(\frac{x^4}{(1-x-y)^2}+\frac{x^5y}{(1-x-y)^3}\right)\left(\log\frac{x}{1-x}+\log\frac{y}{1-y}\right)\right\}dx\,dy\ldots.(9).$$

$$(2f+3i)\iiint\frac{x^2y^3}{\Delta^2}=(2f+3i)\iint\left\{\frac{x^2}{y(1-x-y)}+\frac{2x^3}{(1-x-y)^2}\right.$$
$$\left.+2\left(\frac{x^3}{(1-x-y)^2}+\frac{x^4y}{(1-x-y)^3}\right)\left(\log\frac{x}{1-x}+\log\frac{y}{1-y}\right)\right\}dx\,dy\ldots\ldots(10).$$

$$(f+2i)\iiint\frac{x^2y^2}{\Delta^2}=(f+2i)\iint\left\{\frac{x}{y(1-x-y)}+\frac{2x^2}{(1-x-y)^2}\right.$$
$$\left.+2\left(\frac{x^2}{(1-x-y)^2}+\frac{x^3}{(1-x-y)^3}\right)\left(\log\frac{x}{1-x}+\log\frac{y}{1-y}\right)\right\}dx\,dy\ldots\ldots(11).$$

$$i\iiint\frac{xy^3}{\Delta^2}=i\iint\left\{\frac{x^2}{y(1-x-y)}+\frac{x^3}{(1-x-y)^2}\left(\log\frac{x}{1-x}+\log\frac{y}{1-y}\right)\right\}dx\,dy$$
$$\ldots\ldots\ldots(12).$$

$$i\iiint\frac{xy^2}{\Delta^2}=i\iint\left\{\frac{1}{y(1-x-y)}+\frac{2x}{(1-x-y)^2}+2\left(\frac{x}{(1-x-y)^2}+\frac{x^2y}{(1-x-y)^3}\right)\right.$$
$$\left.\times\left(\log\frac{x}{1-x}+\log\frac{y}{1-y}\right)\right\}dx\,dy\ldots\ldots\ldots(13).$$

This is sufficient to indicate the method of solving the general question. In the case of the equilateral triangle, the labour is much lightened; for, in this instance, $a=b=c=l=m=n=1$, and accordingly $e=2$. $f=6$, $g=3$, $h=6$, $i=3$, $k=6$, $p=1$. We are therefore able, by addition and subtraction of coefficients, to form the following integral:—

$$\iint\left\{\frac{5+15x^2-6x^3+3xy-15y}{1-x-y}-\frac{3-12x+18x^2-12x^3}{y(1-x-y)}\right.$$
$$+\frac{6x+24x^2-28\frac{1}{2}x^3+12xy+24x^4+24x^3y+6x^2y^2-30xy^2}{(1-x-y)^2}+\frac{9x^4y}{(1-x-y)^3}$$
$$+\frac{12x+48x^2-78x^3+48x^4+22xy-60xy^2+12x^2y^2+48x^3y-12x^4y}{(1-x-y)^2}\log\frac{x}{1-x}$$
$$+\frac{48x^3+12x^2y+24x^2y^2+48x^4y^2+12x^3y^3-60x^2y^3+48x^5y-84x^4y}{(1-x-y)^3}\log\frac{x}{1-x}$$
$$\left.+\frac{18x^3(1-x-y+xy)^2}{(1-x-y)^4}\log\frac{x}{1-x}\right\}dx\,dy,\text{ of which the first term becomes}$$

$$\int\{15-3x+(18x+12x^2-6x^3-10)\log(1-1/x)\}\,dx,$$

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and, as this involves the logarithm of a negative quantity, it is difficult to see how further integration is possible. Moreover, to show that there is little hope of eliminating these awkward quantities, it may be observed that the four first terms reduce to

$$\int\{27-129x+71\tfrac{1}{2}x^2-57x^3-57x^4/(1-x) - [75x-309x^2+390x^3-192x^4+24x^5+3x^6/(1-x)]\log(-x) + [66x-300x^2+390x^3-192x^4+24x^5-10]\log(1-x)\}\,dx.$$

Yet there is a good *à priori* reason why the result should be definite.

8428. (By L. J. Rogers, B.A.)—Prove that, if $\alpha, \beta \ldots$ be any positive quantities, and $s_r \equiv \alpha^r + \beta^r + \ldots$, then $s_m s_r > s_n s_p$, provided

$$m > n > p > r \quad \text{and} \quad m+r = n+p.$$

Solution by R. F. Davis, M.A.; W. J. Barton, M.A.; *and others.*

$$\begin{aligned} s_m s_r - s_n s_p &= (a^m b^r + a^r b^m - a^p b^n - a^n b^p) + \text{similar terms} \\ &= a^r b^r \{a^{m-r} + b^{m-r} - a^{p-r} b^{n-r} - a^{n-r} b^{p-r}\} + \ldots \\ &= a^r b^r (a^{p-r} - b^{p-r})(a^{n-r} - b^{n-r}) + \ldots, \end{aligned}$$

for $$(p-r) + (n-r) = n+p-2r = m-r,$$

$$= \text{a positive quantity.}$$

For, a, b being positive, the two binomial factors are either both positive or both negative according as $a >$ or $< b$.

8664. (By J. Brill, M.A.)—O is the mean centre of the points A, B, C for multiples l, m, n respectively. Prove that, if $OA = \xi$, $OB = \eta$, $OC = \zeta$, and $2\sigma = l\xi + m\eta + n\zeta$, the area of the triangle ABC is

$$\frac{l+m+n}{lmn}\{\sigma(\sigma-l\xi)(\sigma-m\eta)(\sigma-n\zeta)\}^{\frac{1}{2}}.$$

Solution by Professor Neuberg.

On sait que l, m, n sont proportionnels aux aires des triangles OBC, OCA, OAB. Donc, si α, β, γ désignent les angles BOC, AOC, BOA, et S l'aire ABC, on a les égalités

$$\tfrac{1}{2}\eta\zeta\sin\alpha = \frac{Sl}{l+m+n}, \quad \tfrac{1}{2}\zeta\xi\sin\beta = \frac{Sm}{l+m+n}, \quad \tfrac{1}{2}\xi\eta\sin\gamma = \frac{Sn}{l+m+n}.$$

De plus, de $\alpha+\beta+\gamma = 2\pi$, on tire

$$\sin\alpha = -\sin\beta(1-\sin^2\gamma)^{\frac{1}{2}} - \sin\gamma(1-\sin^2\beta)^{\frac{1}{2}},$$

ou sous forme rationnelle

$$4\sin^2\alpha\sin^2\beta\sin^2\gamma = (\sin\alpha+\sin\beta+\sin\gamma)(\sin\alpha+\sin\beta-\sin\gamma)\ldots$$

En éliminant les angles auxiliaires α, β, γ, et en résolvant l'égalité obtenue par rapport à S, on aura la formule proposée.

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8460. (By J. O'Byrne Croke, M.A. Suggested by Quest. 8430.)—Those sectors of an ellipse are equal in area which are included between radii m and n, and a diameter o, drawn respectively parallel to two tangents from a point P, and their chord of contact.

Solution by R. Curtis, M.A.; Prof. Nilkantha Sarkar, M.A.; *and others.*

For a circle the proposition is evidently true: the radii drawn parallel to a pair of tangents must make equal angles with the diameter parallel to the chord of contact, and therefore the sectors included between the radii and the diameter are equal.

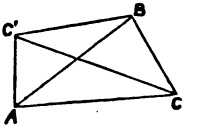
It may be proved for an ellipse by orthographically projecting the ellipse into a circle.

8415. (By B. Hanumanta Rau, B.A.)—If A′, B′, C′ and A, B, C are the angles of a triangle, prove that

$$\left(\frac{\sin B'}{\sin B}\right)^2 + \left(\frac{\sin C'}{\sin C}\right)^2 - 2\frac{\sin B'.\sin C'}{\sin B\,.\,\sin C}\cos(A+A')$$
$$:\left(\frac{\sin C'}{\sin C}\right)^2 + \left(\frac{\sin A'}{\sin A}\right)^2 - 2\frac{\sin C'.\sin A'}{\sin C\,.\,\sin A}\cos(B+B') = \sin^2 A : \sin^2 B.$$

Solution by Isabel Madison; W. J. Barton, M.A.; *and others.*

On the side AB of the triangle ABC, construct a triangle BC′A, so that the angle ABC′ = B′, the angle BAC′ = A′, and consequently the angle AC′B = C′; and join CC′. Now



$$CC'^2 = C'A^2 + CA^2 - 2C'A\,.\,CA\cos(A+A'),$$

and (if AB = c),

$$C'A = c\frac{\sin B'}{\sin C'}, \quad CA = c\frac{\sin B}{\sin C},$$

therefore
$$\frac{CC'^2}{c^2} = \frac{\sin^2 B'}{\sin^2 C'} + \frac{\sin^2 B}{\sin^2 C} - 2\frac{\sin B'.\sin B}{\sin C'.\sin C}\cos(A+A'),$$

$$\therefore \quad \frac{CC'^2}{c^2}\cdot\frac{\sin^2 C'}{\sin^2 B} = \frac{\sin^2 B'}{\sin^2 B} + \frac{\sin^2 C'}{\sin^2 C} - 2\frac{\sin B'.\sin C'}{\sin B\,.\,\sin C}\cos(A+A').$$

Similarly, since $CC'^2 = C'B^2 + CB^2 - 2C'B\,.\,CB\cos(B+B')$,

$$\frac{CC'^2}{c^2}\cdot\frac{\sin^2 C'}{\sin^2 A} = \frac{\sin^2 C'}{\sin^2 C} + \frac{\sin^2 A'}{\sin^2 A} - 2\frac{\sin C'.\sin A'}{\sin C\,.\,\sin A}\cos(B+B');$$

hence the stated ratio

$$= \frac{CC'^2}{c^2}\cdot\frac{\sin^2 C'}{\sin^2 B} : \frac{CC'^2}{c^2}\cdot\frac{\sin^2 C'}{\sin^2 A} = \sin^2 A : \sin^2 B.$$

[Since $c^2b'^2 + b^2c'^2 - 2bc\,b'c'\cos(A+A')$

$= c^2b'^2 + b^2c'^2 + 8\Delta\Delta' - \frac{1}{2}(b^2+c^2-a^2)(b'^2+c'^2-a'^2)$

$= 8\Delta\Delta' - \frac{3}{4}(a^2a'^2 + b^2b'^2 + c^2c'^2) + \frac{1}{4}(aa' + bb' + cc')^2$

a symmetrical expression, therefore the required result follows.]

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8450. (By Rev. T. R. Terry, M.A.)—The sum of the squares of the coefficients in the expansion of $(1+x+x^2)^n$, where n is a positive integer, is $\Sigma \frac{2n!}{r!\,r!\,(2n-2r)!}$, where r may have any value from 0 to n inclusive.

Solution by R. F. Davis, M.A.; W. W. Taylor, M.A.; *and others.*

The required sum = coefficient of x^{2n} in the expansion of $(1+x+x^2)^{2n}$ the general term of which is $\frac{2n!}{r!\,(2n-r)!}x^{2r}(1+x)^{2n-r}$, where r may have any value from 0 to $2n$ inclusive, but of which, however, values greater than n are not concerned. The coefficient of x^{2n-2r} in the expansion of $(1+x)^{2n-r}$ is $\frac{(2n-r)!}{(2n-2r)!\,r!}$. Hence, &c.

8538. (By Rev. A. F. Torry, M.A.)—Prove that (1) the tangent to $27ay^2 = 4(x-2a)^3$ where it meets $y^2 = 4ax$ is normal to both curves; and (2) the ratio in which this straight line is divided by the curves is

$$8a - \tfrac{1}{2}(7a) : \tfrac{1}{2}(7a) - 2a = 3 : 1.$$

Solution by J. Young, M.A.; Rev. T. Gallibrs, M.A.; *and others.*

1. The equation determining the points of intersection is $27a^2x = (x-2a)^3$, whose roots are $-a$, $-a$, $8a$; the latter only gives a real value for y, namely $4a\sqrt{2}$. The tangent to the first curve at this point is found in the usual way to be $y = (x-4a)\sqrt{2}$. Combining this equation with the equation of the curve, we have the cubic $27a(x-4a)^2 = 2(x-2a)^3$, two of whose roots being $8a$, the remaining root is found to be $7a/2$, and the corresponding value for y is $-a/\sqrt{2}$. The same line meets the other curve in the point whose coordinates are $2a$, $-2a\sqrt{2}$. The equations to the tangents to the curves at these two points are $2x + 2\sqrt{2}y = 5a$, $2x + 2\sqrt{2}y = -4a$, and these two lines are parallel to one another and perpendicular to the tangent to the first curve at the common point.

2. The required ratio is easily found to be as stated.

8545. (By D. Edwardes.)—A uniform rod moves in a vertical plane under the action of gravity, with its lower extremity upon a smooth horizontal plane, and its upper extremity constrained to move upon a smooth curve. If the rod starts from rest, prove that it reaches the horizontal position in the shortest time when the curve is an arc of an ellipse of eccentricity $\frac{1}{2}(\sqrt{3})$ and latus rectum equal to half the length of the rod; and, assuming that the rod in its initial position is not to cut the curve, show (2) that there are two ellipses satisfying the condition, if the tangent of the inclination of the rod to the horizon in its initial position is less than $\sqrt{2}$, the time being the same for both curves.

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Solution by Professors SIRCOM, NASH, *and others.*

It is evident that the rod cannot fall faster than it would if its upper extremity were entirely free, but in that case the centre of gravity would describe a vertical line, and therefore its upper extremity an ellipse, by the principle of the elliptic compasses. The curve of quickest descent will therefore be the ellipse $y^2 + 4(x-c)^2 = 4a^2$, if $2a$ be the length of the rod, with eccentricity $\frac{1}{2}\sqrt{3}$, and latus rectum a, the major axis being vertical. The time down the curve, being given by the equation of energy, depends only on the ordinate of the centre of gravity, and therefore on y of the curve; but for any initial value of y there are two values of x, for one of which the rod moves on the concave, for the other on the convex side of the curve, for both of which the equation of energy and time of descent will be the same; or, which is the same thing, the conditions are satisfied by either of the curves $y^2 + 4(x \pm c)^2 = 4a^2$, where $c = a\cos\alpha$, α being the initial inclination of the rod, provided the rod do not cut the curve $y^2 + 4(x+c)^2 = 4a^2$. It will be a tangent to the curve at its upper extremity if $\tan\alpha = -\dfrac{dy}{dx} = \dfrac{4x+c}{y}$, which, when $x = 0$, $y = 2a\sin\alpha$, becomes $2\cot\alpha$; therefore $\tan\alpha = \sqrt{2}$, and if $\tan\alpha$ be greater than this value the rod will cut the curve.

The result may be easily obtained by the Calculus of Variations from the equation of energy, but this seems unnecessary.

8547. (By A. GORDON.)—Prove that (1) the locus of points for which the sum of the squares of the normals to the parabola $y^2 = lx$ is constant ($= c^2$) is the ellipse $3y^2 + (x+l)^2 = \frac{3}{2}l^2 + c^2$; and (2) the corresponding locus for the general equation of the second degree $(x, y, 1)^2 (a, b, c, f, g, h)$ is a conic.

Solution by J. YOUNG, M.A.; Rev. T. GALLIERS, M.A.; *and others.*

1. The equation for determining the ordinates of normals through the point $\alpha\beta$ is $2y^3 + (l^2 - 2l\alpha)\,y - l^2\beta = 0$, and the required locus is

$$\Sigma(x-\alpha)^2 + \Sigma(y-\beta)^2 = c^2, \text{ or } 1/l^2\,\Sigma(y^4) + (l-2\alpha)/l\,\Sigma y^2 + 3\alpha^2 + 3\beta^2 = c^2.$$

Now $$\Sigma(y) = 0, \quad \Sigma(y^2) = -2\Sigma(y_1 y_2) = -(l^2 - 2l\alpha),$$

$$\Sigma(y^4) = 2\left\{\Sigma(y_1 y_2)\right\}^2 = \tfrac{1}{2}(l^2 - 2l\alpha)^2.$$

Substituting these values, $-\frac{1}{2}(l-2\alpha)^2 + 3\alpha^2 + 3\beta^2 = c^2$. Changing α, β into x, y, and simplifying, the given equation is obtained.

2. The general equation of the second degree being reduced to the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, the equation for determining the ordinates is

$$c^4y^4 - 2b^2c^2\beta y^3 + (a^2\alpha^2 + b^2\beta^2 - c^4)\,b^2y^2 + 2b^4c^2\beta y - b^6\beta^2 = 0,$$

with a corresponding equation in x; and the equation of the required locus $\Sigma(x-\alpha)^2 + \Sigma(y-\beta)^2 = c^2$ will be found to involve powers of α, β, as far as the second degree only.

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APPENDIX I.

UNSOLVED QUESTIONS BY W. J. C. SHARP, M.A.

***** Where references to previous volumes are appended, the Questions have been previously solved, but other solutions are desired.**

5640. SALMON says (*Higher Plane Curves*, p. 98) that the curve parallel to a given curve may be obtained (1) as the envelope of a circle of given radius whose centre moves on the given curve; (2) as the envelope of the parallel to the tangent to the given curve drawn at a constant distance. Prove that these processes are equivalent.

5755. If, through any point of inflexion O on an n-ic curve, there be drawn three straight lines meeting the curve again in $A_1, A_2 \dots A_{n-1}$, $B_1, B_2 \dots B_{n-1}$, $C_1, C_2 \dots C_{n-1}$ respectively, prove that every curve of the n^{th} degree through the $3n-2$ points $O_1, A_1, A_2 \dots A_{n-1}$, $B_1 \dots B_{n-1}$, $C_1 \dots C_{n-1}$ will have O for a point of inflexion.

5806. Find (1) the law of distribution of mass such that the catenary (under the action of gravity) may be a given curve: for example, the portion of an ellipse cut off by its horizontal major axis; and (2), supposing the mass supplied by a fringe of uniform separate threads, find the locus of the ends, both generally and in the particular case.

6271. Of all the lines whose satellites, with respect to a nodal cubic, pass through the node, prove that each passes through one of four fixed points on the harmonic polar of the real inflexion.

6371. If a uniform wire, of such a nature that the resistance to bending varies as the curvature, be suspended, without torsion or extension of the mean fibre, from two points; show that it will hang in a catenary identical with that in which a uniform string of the same length would hang, but that the tension along the wire will differ by a constant quantity from that along the string.

6686. If a bar naturally curved be strained, the bending moment at a point whose natural curvature is r^{-1} and strained curvature ρ^{-1}, is

$$E(\rho^{-1} - r^{-1})\{I + Ar^{-2} + Br^{-4} + \&c.\}.$$

6903. Find the law of density, a function of the depth, that the centre of pressure of a semicircle, just immersed with its diameter horizontal, may divide the vertical radius in the proportion of $m : n$. [See Vol. XLIV.,

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p. 94; Vol. XLV., p. 164; the solution in Vol. XLV. is not such as the author wished for, and is based on a misunderstanding which may be avoided by the resetting the question as given above.]

7174. The result of translating a system of vectors β along a vector α, and rotating them through an angle twice the angle of the quaternion q about its axis, is $q(\alpha+\beta)q^{-1}$ or $\alpha+q\beta q^{-1}$, according to the order in which the operations are performed. Show that the operations are identical if α be the axis of q. [This is one of the advantages of Dr. BALL's Theory of Screws.]

7178. If three concyclic foci of a bicircular quartic, or circular cubic, be given, and also a tangent and its point of contact, determine the curve.

7203. In the late Prof. CLIFFORD's Quest. 6009 (*Reprint*, Vol. XXXIII., p. 59),—as to what is the form of a kite string,—assuming that the force of the wind is a function of the height, show how this function may be determined, so that the string shall assume a given form. (Vol. XXXIX., p. 99.)

7209. If a circle be described upon a common tangent to two given circles as diameter, show that it cuts the line of centres in points which are inverse to each other with respect to both circles. Hence prove that confocal conics cut at right angles.

7270. Eliminate a, b, c, a', b', c' from

$$\begin{array}{r|c|l} a+b+c=d & a'+b'+c'=d' & aa'+bb'+cc'=e', \\ ab+bc+ca=s & a'b'+b'c'+c'a'=s' & a'bc+ab'c+abc'=\theta \\ abc\quad=\delta & a'b'c'\quad=\delta' & ab'c'+a'bc'+a'b'c=\theta'. \end{array}$$

7318. Show that the tangents from the point (x_1, y_1, z_1), on the non-singular cubic $x^3+y^3+z^3+6mxyz=0$, touch at points of intersection of the conics $x_1x^2+y_1y^2+z_1z^2=0$ and $\frac{x_1}{x}+\frac{y_1}{y}+\frac{z_1}{z}=0$; and hence determine (1) the equation to the satellite of the line $ax+\beta y+\gamma z=0$; (2) the quartic curve which meets the cubic at points of contact of the tangents drawn from the intersections of $ax+\beta y+\gamma z=0$ with the curve.

7361. In a circular cubic, prove that (1) the directrices, or chords of contact, corresponding to concyclic foci, pass through the centre of the focal circle, which is the point of contact of one of the tangents parallel to the real asymptote; also (2) the four centres lie at the intersections of the cubic and a rectangular hyperbola which has an asymptote in common with the cubic. [Vol. XLVI., p. 114. This solution is quite different from the one desired, though apparently perfectly valid.]

7394. If a quartic equation represent four straight lines, its Hessian represents the same four lines and an imaginary conic. Explain the geometrical signification of this.

7400. Of the twenty-four lines which touch two of the circles inscribed and escribed to a triangle, eighteen are represented by the sides; construct the other six.

7410. If N : D be a fraction in its lowest terms, and $D \equiv 2^h.5^k.a^l.b^m.c^n \ldots$, where a, b, c, &c. are prime numbers, the equivalent decimal will

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consist of h or k non-recurring figures (according as h or k is greatest), and of a recurring period, the number of figures in which is a measure of $a^{l-1}(a-1) \cdot b^{m-1}(b-1) \cdot c^{n-1}(c-1) \ldots$ (Vol. XLI., p. 113).

7418. (Suggested by Quest. 7227.) Find in how many ways p sets of n things, the individuals of each of which are marked $1 \cdot 2 \ldots n$, can be permuted, so that no two individuals marked with the same number shall occupy the same position in any two sets.

7447. If a rigid body have one point fixed, show that the *vis-vivâ* of the motion is $\frac{1}{2}[A\omega_1^2 + B\omega_2^2 + C\omega_3^2 - 2D\omega_2\omega_3 - 2E\omega_3\omega_1 - 2F\omega_1\omega_2]$, where A, B, C, D, E, and F are the moments and products of inertia, and ω_1, ω_2, ω_3 the angular velocities about any rectangular axis fixed in the body and passing through the fixed point.

7477. If a curve be defined by a linear relation among the distances of a point upon it from any number of fixed points, each of these is a focus.

7528. Prove that $$\frac{(1-x^{n+1})(1-x^{n+2}) \ldots (1-x^{n+\theta})}{(1-x)(1-x^2) \ldots (1-x^{\theta})}$$ is equal to $$\frac{(1-x^{\theta+1})(1-x^{\theta+2}) \ldots (1-x^{n+\theta})}{(1-x)(1-x^2) \ldots (1-x^n)},$$ and hence deduce HERMITE's Law of Reciprocity (SALMON's *Modern Higher Algebra*, p. 125.) (Vol. XLV., p. 43.)

7627. If two tetrahedrons be such that the intersections of corresponding faces lie in the same plane, the lines joining corrresponding vertices, and the planes through corresponding edges, all meet in a point, and conversely.

7656. If ABCD be a tetrahedron, p_1, p_2, p_3, p_4 the perpendiculars from the vertices upon the opposite faces, then, denoting by (AB), &c., the dihedral angles between the faces which intersect in AB, &c., prove that (1)

$$\sin(AB) : \sin(BC) : \sin(CA) : \sin(AD) : \sin(BD) : \sin(CD)$$
$$= \frac{AB}{p_1 p_2} : \frac{BC}{p_2 p_3} : \frac{CA}{p_3 p_1} : \frac{AD}{p_1 p_4} : \frac{BD}{p_2 p_4} : \frac{CD}{p_3 p_4};$$

and (2) the equation to the sphere described about the tetrahedron may be written (SALMON's *Solid Geometry*, p. 171),

$$ab \sin(ab) \cdot xy + bc \sin(bc) \cdot yz + \&c. = 0.$$

[The solution in Vol. XLII., p. 54, is incomplete.]

7688. Find the curve in which a kite-string will hang, when acted on by a uniform wind blowing in a direction inclined to the horizon. [The solution in Vol. XLII., p. 100, is incorrect.]

7715. Two bags contain, the one n white balls and the other n black; and each set is numbered $1 \cdot 2 \ldots n$. The balls are drawn in pairs, one from each bag; find the chance, at any stage, that no pair has been drawn consisting of balls bearing the same number; and show that, as the number drawn increases, this approaches to e^{-1}.

7761. A flexible string is suspended slackly from two fixed points, and acted upon by a uniform horizontal wind, blowing in a direction making any angle with the horizontal projection of the line joining the

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points. Find the curve in which the string hangs, and the tension at any point. [The solution in Vol. XLII., p. 100, is incorrect.]

7834. If lines be drawn from the vertices of any tetrahedron ABCD through the same point P, to meet the opposite faces in A′, B′, C′, D′ respectively: show that

$$\frac{PA'}{AA'} + \frac{PB'}{BB'} + \frac{PC'}{CC'} + \frac{PD'}{DD'} = 1. \quad \text{(Vol. XLV., p. 41.)}$$

7847. If we denote the operation $l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}$, where l, m, n are direction cosines, by $\frac{d}{dh}$, so that $\frac{d}{dh}$ is the differential coefficient with respect to an axis (CLERK MAXWELL'S *Electricity*, 2nd ed., p. 180); show that it is an invariant symbol of operation.

7856. Construct an octahedron such that it shall have two parallel faces equal and similar to each of the faces of a given tetrahedron, and show that its volume is four times that of the tetrahedron.

7882. Prove that the equation

$$\frac{d^2y}{dx^2} + \phi(x)\frac{dy}{dx} + \psi(x)\, y = 0$$

is reducible to an equation with constant coefficients, if

$$\frac{d}{dx}\left\{\frac{\psi'(x) + 2\phi(x)\psi(x)}{[\psi(x)]^{\frac{3}{2}}}\right\} = 0.$$

7899. If lines be drawn from the vertices of a triangle through any point on a circumscribed conic to meet the opposite sides; prove that the axis of homology of the triangle formed by the connectors of the intersections and the original triangle will pass through a fixed point, which is on each of the fourth harmonics to the tangents at the vertices.

7918. Prove that

$$a(a+1)\ldots(a+n-1) + \frac{n}{1}a(a+1)\ldots(a+n-2)\,.\,b$$

$$+ \frac{n(n-1)}{1\,.\,2}a(a+1)\ldots(a+n-3)\;1\,.\,b(b+1) + \&c. + b(b+1)\ldots(b+n-1)$$

$$= (a+b)(a+b+1)\ldots(a+b+n-1),$$

and verify the result from the theory of Combinations.

7942. If P be the centroid of three weights a, b, c at the points A, B, C, prove that

$$\frac{a\,.\,AP}{\sin BPC} = \frac{b\,.\,BP}{\sin CPA} = \frac{c\,.\,CP}{\sin APB}.$$

[The solution in Vol. XLIV., p. 50., is not satisfactory.]

7954. In a triangle ABC, if p_1 be the perpendicular from A upon BC, r the radius of the in-circle, and r_1 that of the e-circle touching BC, show that $\frac{1}{r} - \frac{1}{r_1} = \frac{2}{p_1}$; also show that the same equation holds if p_1 be the perpendicular from the vertex A of a tetrahedron ABCD upon the opposite face, r the radius of the in-sphere, and r_1 that of the sphere

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touching BCD and the other faces produced. (This may easily be proved without assuming the values of r, &c.) [Vol. XLIII., p. 71; Vol. XLVI., p. 37. The solution in Vol. XLIII. is imperfect, and not according to the directions.]

7980. If AA′, BB′, &c. be concurrent lines drawn from the vertices of any tetrahedron ABCD to meet the opposite faces in A′, B′, &c., show that

triangle A′BC : triangle A′DB = triangle B′AC : triangle B′DA,
triangle A′BD : triangle A′CD = triangle D′BA : triangle D′AC, &c.

8019. If the weights of three bodies, whose centroids are at the vertices of a triangle, are connected by a homogeneous relation of the nth order, the locus of the centroid of the system is a plane n-ic curve; and if those of four bodies, whose centroids occupy the vertices of a tetrahedron, are connected by such a relation, the locus is an n-ic surface.

8033. (Suggested by Question 7536.) If $3n-1$ points be given on a plane cubic and an $(n-\nu)$-ic curve be described through any $3n-3\nu-1$ of these, and any $(\nu+1)$-ic curve through the remaining intersection of this with the cubic and the other 3ν given points; prove (1) that this will cut the cubic in two additional points, the line joining which passes through the single point residual to the $3n-1$ given points; (2) enunciate the reciprocal proposition.

8058. Prove that every unicursal twisted cubic curve is the partial intersection of three quadric surfaces, each pair of which has a common generator.

8098. Show that, if p be any prime and n an integer prime to p, $x^{n(p-1)}+x^{n(p-2)}+x^{n(p-3)}+\ldots+1$ is divisible by $x^{p-1}+x^{p-2}+x^{p-3}+\ldots+1$; and, if n be even, also by

$$x^{p-1}-x^{p-2}+x^{p-3}+\ldots+(-1)^{p-1};$$

and that, when n is odd, this last quantity divides

$$x^{n(p-1)}-x^{n(p-2)}+x^{n(p-3)}+\ldots+(-1)^{p-1}.$$

8141. If Δ be the discriminant of a quadric $U=0$; show that Δ is positive, negative, or zero, according as two distinct generators, no real generators, or two coincident generators can be drawn through each point on the surface.

8175. Dr. Boole notices a remarkable duality which exists in partial differential equations (*Differential Equations*, p. 366, ed. I.), and gives formulæ for the transformation of equations of the first and second order with two independent variables. Give the corresponding formulæ for any number of variables; and show that, if a partial differential equation be linear in the first minors of the Hessian of the dependent variable, except a term involving the Hessian, it can be transformed into a linear form.

8207. If A be the angle contained in the half of a small circle of a sphere, angular radius $a/2$, by arcs of lengths b and c; *i.e.*, if ABC be a spherical triangle having the angle $A=B+C$; show that (1) $\cos b+\cos c=1+\cos a$; (2) $\cos A=-\tan(b/2)\tan(c/2)$; and (3) hence deduce Euclid I. 47, and III. 31.

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8237. If ${}_nC_r$ denote the number of combinations r together which can be formed out of n things; show, from *à priori* considerations, that

$$ {}_nC_r = {}_nC_{n-r}, \quad {}_nC_r = \frac{n}{r} {}_{n-1}C_{r-1}, \quad {}_nC_r = {}_{n-1}C_r + {}_{n-1}C_{r-1} \ldots\ldots (1, 2, 3), $$

or, more generally, ${}_nC_r = {}_{n-p}C_r + {}_pC_1 \, {}_{n-p}C_{r-1} + {}_pC_2 \, {}_{n-p}C_{r-2} +$ &c.

8263. Find y, so that

$$ \frac{\int_0^x \phi(x)\, y dx}{\int_0^x \psi(x)\, y dx} = \chi(x), $$

where $\phi(x)$, $\psi(x)$, and $\chi(x)$ are integral and rational functions of x. The value of y may be found either by a differential equation of the first order, or by one of the nth order, where n is the order of the highest of the two functions ϕ and ψ. Hence, show that the equation

$$ [\phi(x) - \chi(x)\, \psi(x)] \frac{d^n y}{dx^n} - [\phi'(x) - \chi(x)\, \psi'(x)] \frac{d^{n-1} y}{dx^{n-1}} $$

$$ + [\phi''(x) - \chi(x)\, \psi''(x)] \frac{d^{n-2} y}{dx^{n-2}} + \&c. + [C - C' \chi(x)] = 0 $$

is always integrable.

8296. If a binary quantic be an exact pth power of another such power, prove that (1) its Hessian is a multiple of the lower quantic raised to the power of $2p-2$ and its Hessian; and (2) whenever a binary quantic measures its own Hessian, it is an exact power of another of a lower order.

8457. If E be the spherical excess of a triangle whose sides are a, b, c; show (1) that

$$ \sin \tfrac{1}{4} \mathrm{E} = \frac{\sin \tfrac{1}{2} s \sin \tfrac{1}{2} (s-a) \sin \tfrac{1}{2} (s-b) \sin \tfrac{1}{2} (s-c)}{\cos \tfrac{1}{2} a \cos \tfrac{1}{2} b \cos \tfrac{1}{2} c}, $$

where $2s = a + b + c$; and hence (2) deduce the ordinary formula for the area of a plane triangle.

8534. Solve the equations

$$ u_n = u_{n-1} + p\,(n-1)\, u_{n-2}, \quad u_n = u_{n-1} + p\,(n-1)\,(n-2)\, u_{n-2}, $$
$$ u_n = u_{n-1} + p\,(n-1)\,(n-2)\,(n-3)\, u_{n-2}. $$

8566. If P_n denote the number of partitions, without repetitions, of a number n, and Q_n the number into odd parts; prove that (1) $P_n = Q_n + Q_{n-2} P_1 + Q_{n-4} P_2 +$ &c., and (2) the same formula holds if P_n and Q_n denote the numbers of partitions with repetitions.

8622. Prove that (1) no two curves which are defined by linear relations between the focal distances from any number of common foci, except confocal conics, can cut at right angles; and (2) in particular two circular cubics or bicircular quartics which have a common focal circle and their foci common upon it cannot cut orthogonally.

8733. If the $2n$ points in which a conic cuts an n-ic curve be joined two and two by n straight lines, the other $n(n-2)$ points in which these lines intersect the n-ic lie upon an $(n-2)$-ic curve. Hence the tangentials

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of the n points in which a straight line cuts an n-ic lie upon an $(n-2)$-ic, which may be called the satellite of the line. (Vol. XLVI., p. 97.)

8939. If $S \equiv ax^2 + by^2 + cz^2 + dw^2 + 2lyz + 2mzx + 2nxy + 2pxw + 2qyw + 2rzw = 0$ be the equation to a quadric, show that (1) the line joining the points (x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2) will meet the surface in two real, two coincident, or two imaginary points, according as $S_1S_2 - P_{1.2}^2$ is positive, zero, or negative, where $P_{1.2} = 0$ is the condition that one of the points should lie on the polar plane of the other; and (2) express the same condition in terms of the tangential equation to the quadric.

8940. If

$$S \equiv ax^2 + by^2 + cz^2 + dw^2 + 2lyz + 2mzx + 2nxy + 2pxw + 2qyw + 2rzw,$$

and $$P_{1.2} \equiv ax_1x_2 + by_1y_2 + cz_1z_2 + dw_1w_2 + l(y_1z_2 + y_2z_1) + \&c.;$$

show that $$S_1S_2S_3 + 2P_{1.2}P_{2.3}P_{3.1} - S_1P_{2.3}^2 - S_2P_{3.1}^2 - S_3P_{1.2}^2$$

$$\equiv A \begin{vmatrix} y_1, & y_2, & y_3 \\ z_1, & z_2, & z_3 \\ w_1, & w_2, & w_3 \end{vmatrix}^2 + \ldots + 2L \begin{vmatrix} z_1, & z_2, & z_3 \\ w_1, & w_2, & w_3 \\ x_1, & x_2, & x_3 \end{vmatrix} \begin{vmatrix} w_1, & w_2, & w_3 \\ x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \end{vmatrix} + \&c.,$$

where A, u are the first minors of the discriminant of S.

8941. Prove that the conditions that the binary quantic $(a, b, c \ldots \mathfrak{x}, y)^n$ should be reducible to a binomial form, are

$$\left|\begin{matrix} a, & b, & c, & d \ldots \\ b, & c, & d, & e \ldots \\ c, & d, & e, & f \ldots \end{matrix}\right| = 0.$$

[This is a generalisation of the catalecticant of the quartic; those of quantics of higher order admit of similar extension.]

8942. The polar triangles of the spherical triangles, determined by the four solid angles of a tetrahedron, exactly occupy the surface of the sphere. Construct these, and hence prove the above.

8943. The angle between the great circles bisecting two angles of a spherical triangle, which is subtended by the third side, is the supplement of the angle contained by the chords of the corresponding arcs of the polar triangle.

8944. Form the equation to the two tangent planes, which can in general be drawn to a given quadric, so as to contain a given straight line.

8945. Prove the following formulæ:—(1) For all values of n,

$$2^{n-1}\cos\theta\cos 2\theta \ldots \cos n\theta = \cos\frac{n(n+1)}{2}\theta + \frac{\sin(n-1)\theta}{\sin\theta}\cos\frac{n(n-1)}{2}\theta + \frac{\sin(n-1)\theta\sin(n-2)\theta}{\sin\theta \cdot \sin 2\theta}\cos\frac{n(n-3)}{2}\theta + \&c.;$$

(2) for n odd,

$$2^{n-1}(-1)^{\frac{1}{2}(n-1)}\sin\theta\sin 2\theta \ldots \sin n\theta = \sin\frac{n(n+1)}{2}\theta - \frac{\sin(n-1)\theta}{\sin\theta}\sin\frac{n(n-1)}{2}\theta + \frac{\sin(n-1)\theta\sin(n-2)\theta}{\sin\theta \cdot \sin 2\theta}\sin\frac{n(n-3)}{2}\theta - \&c.;$$

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for n even,

$$2^{n-1}(-1)^{\frac{1}{2}n}\sin\theta\sin 2\theta\ldots\sin n\theta = \cos\frac{n(n+1)}{2}\theta$$

$$-\frac{\sin(n-1)\theta}{\sin\theta}\cos\frac{n(n-1)}{2}\theta+\frac{\sin(n-1)\theta\sin(n-2)\theta}{\sin\theta\,.\,\sin 2\theta}\cos\frac{n(n-3)}{2}\theta-\&c.$$

8946. Show that

$$\begin{vmatrix} x, & -a, & -a, \ldots & -a, & -a \\ b, & x, & -a, \ldots & -a, & -a \\ b, & b, & x, \ldots & -a, & -a \\ & \ldots & \ldots & \ldots & \\ b, & b, & b, \ldots & x, & -a \\ b, & b, & b, \ldots & b, & b \end{vmatrix} = b(x+a)^n,$$

where the determinant has $n+1$ rows.

8947. The lines of curvature on a developable surface are either tangents or evolutes of the cuspidal edge.

8948. Show that

$$\sin n\theta = \frac{1}{n-1!\,\sin^n\theta}\left(\sin^2\theta\frac{d}{d\theta}\right)^{n-1}(\sin^2\theta),$$

and

$$\cos n\theta = \frac{1}{n-1!\,\sin^n\theta}\left(\sin^2\theta\frac{d}{d\theta}\right)^{n-1}(\sin\theta\cos\theta).$$

8949. The unicursal quartic

$$\frac{a}{x^2}+\frac{b}{y^2}+\frac{c}{z^2}+\frac{2f}{yz}+\frac{2g}{zx}+\frac{2h}{xy}=0$$

may be derived from the conic $a\xi^2+b\eta^2+c\zeta^2+2f\eta\zeta+2g\zeta\xi+2h\xi\eta=0$ by a double process of reciprocation.

8950. The p-ary n-ic has generally only

$$\frac{(n+1)(n+2)\ldots(n+p-1)}{1\,.\,2\ldots(p-1)}-p^2+1$$

independent invariants, p independent covariants, including the quantic itself, and p independent contravariants.

8951. If n be any whole number, which is not a multiple of 5, show that

(1) $x^{4n}+x^{3n}+x^{2n}+x^n+1$ is divisible by $x^4+x^3+x^2+x+1$,

(2) $x^{4n}+x^{3n}+x^{2n}+x^n+1$,, ,, $x^4-x^3+x^2-x+1$ if n be even,

(3) $x^{4n}-x^{3n}+x^{2n}-x^n+1$,, ,, $x^4-x^3+x^2-x+1$ if n be odd.

8952. If a sextic equation can be reduced to a cubic, *i.e.*, to an equation having three pairs of roots equal in value and opposite in sign, the skew invariant will vanish.

8953. All the inflexional tangents of a quartic curve touch two curves, of classes four and six respectively. If the quartic be unicursal, these curves are only of the second and third classes.

8954. If seven tangents to a cuspidal cubic (or tricuspidal quartic) be given, and a conic be described to touch any four of those, the conic which touches the other three given tangents and the two remaining

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common tangents of the first conic and the curve, will always touch a fixed tangent to the curve.

8955. Interpret the transformation

$$(x, y, z, p, q \text{ into } p, q, px+qy-z, x, y),$$

given in BOOLE's *Differential Equations*, Chapter XV., geometrically, and show that it may be extended to any number of variables. Also give the corresponding transformations of the second differential coefficients in the latter case.

8956. No function of one variable can be more than singly periodic for real values of the variable. A fresh imaginary period may arise for each set of incommensurable imaginaries which is recognised.

8957. If a rational relation subsists between the distances of any point on a curve from two or more fixed points, and involves an odd power of one of them, the point from which this is measured is a focus, *i.e.*, the real intersection of two tangents to the curve from the circular points at infinity.

8958. Show that the differential equation to a uniform flexible surface hanging under the action of gravity and bounding forces is

$$f(y)\,r+\phi(x)\,t=m\,[(1+p^2+q^2)]^{\frac{1}{2}},$$

where $f(y)$ and $\phi(x)$

are arbitrary functions to be determined by the boundary conditions.

8959. If any function of the coefficients of the ternary quantic

$$a_0x^n+n\,(a_1y+b_1z)\,x^{n-1}+\tfrac{1}{2}n\,(n-1)\,(a_2y^2+2b_2yz+c_2z^2)\,w^{n-2}+\&c.$$

vanish when α^n, $\alpha^{n-1}\beta$, $\alpha^{n-1}\gamma$, $\alpha^{n-2}\beta^2$, &c. are substituted for a, a_1, b_1, a_2, &c., the evertant of the function will vanish, and conversely.

8960. The most general expression for a spherical harmonic of order n (an integer) is $\Sigma\,\frac{n!}{k!\,l!\,m!}\,a_{k,l,m}\,x^ky^lz^m$, where $k+l+m=n$ and the coefficients satisfy all the equations of the form

$$a_{k+2,l,m}+a_{k,l+2,m}+a_{k,l,m+2}=0$$

(where $k+l+m=n-2$) which can be formed for different integral values of k, l, and m. Hence, if $(a, b, c, f, g, h\,\Box\,x, y, z)^2$ be a spherical harmonic, $a+b+c=0$.

8961. If a curve in space be denoted by the equations

$$x=(a_0, a_1 \dots a_n\,\Box\,t, 1)^n, \quad y=(b_0, b_1 \dots b_n\,\Box\,t, 1)^n,$$
$$z=(c_0, c_1 \dots c_n\,\Box\,t, 1)^n, \text{ and therefore } w=(d_0, d_1 \dots d_n\,\Box\,t, 1)^n,$$

the curve will be an n-ic, and $\frac{1}{2}m\,(m^2+6m+11-6n)$, independent m-ic surfaces can be drawn through it. The equations to the tangent at t are

$$\begin{vmatrix} (a_0, a_1 \dots a_{n-1}\,\Box\,t, 1)^{n-1} & (b_0, b_1 \dots b_{n-1}\,\Box\,t, 1)^{n-1} & (c_0 \dots \Box\,t, 1)^{n-1} \dots \\ (a_1, a_2 \dots a_n\,\Box\,t, 1)^{n-1} & (b_1, b_2 \dots b_n\,\Box\,t, 1)^{n-1} & \dots \quad \dots \quad \dots \quad \dots \\ x, & y, & z, \quad w \end{vmatrix} = 0,$$

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and that to the osculating plane is

$$\begin{vmatrix} (a_0, a_1 \dots a_{n-2}\mathfrak{J}t, 1)^{n-2} & (b_0 \dots \mathfrak{J}t, 1)^{n-2} & (c_0 \dots \mathfrak{J}t, 1)^{n-2} & (d_0 \dots \mathfrak{J}t, 1)^{n-1} \\ (a_1, a_2 \dots a_{n-1}\mathfrak{J}t, 1)^{n-2} & \dots & \dots & \dots \\ (a_2, a_3 \dots a_n \mathfrak{J}t, 1)^{n-2} & \dots & \dots & \dots \\ x, & y, & z, & w \end{vmatrix} = 0.$$

8962. The solution of a linear equation of differences can always be made to depend upon that of a linear differential equation of the same order, and *vice versâ*.

8963. (Suggested by Quest. 7545.) The points on a right line have a 1 to 1 correspondence with the rays of a pencil in the same plane. Show that the lines drawn through the points parallel to their corresponding rays touch a parabola of which the line is a tangent.

8964. If two vectors be so related that one is a linear and vector function of the other, the coefficients in the expressions for the components of the one in terms of those of the second are connected by six invariant relations.

8965. Show that, if

$$x^n - \mathrm{P}_1 x^{n-1} + \mathrm{P}_2 x^{n-2} - \mathrm{P}_3 x^{n-3} + \&c. = 0$$

be the equation whose roots are the squares of those of

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - p_3 x^{n-3} + \&c. = 0,$$

$$\mathrm{P}_r = \frac{1}{2\,(n-r)\,!}\left\{\frac{p_{n-1}}{p_n}\left(\frac{d}{dp_1} + p_1\frac{d}{dp_2} + p_2\frac{d}{dp_3} + \&c.\right)\right.$$
$$\left. - \left(n\frac{d}{dp_2} + (n-1)\,p_1\frac{d}{dp_3} + (n-2)\,p_2\frac{d}{dp_4} + \&c.\right)\right\}^{n-r}(p_n^2).$$

8966. If ρ, ρ_1, ρ_2, &c. be the radii of curvature at corresponding points on a curve and its successive evolutes, then for a conic

$$45\rho\rho_1\rho_2 = 9\rho^2\rho_3 + 36\rho^2\rho_1 + 40\rho_1^3,$$

and for a parabola $$3\rho\rho_2 = 9\rho^2 + 4\rho_1^2.$$

8967. If $\mathrm{F}(x)$ be a function defined by the equation $\mathrm{F}(x+p) = n\,\mathrm{F}(x)$, where n and p are constants; show that

$$\mathrm{F}(x) = n^{x/p}\left\{\mathrm{C} + \Sigma^{i=\infty}_{i=1}\left(\mathrm{A}_i \cos\frac{2i\pi x}{p} + \mathrm{B}_i \sin\frac{2i\pi x}{p}\right)\right\},$$

and evaluate the constants.

8968. If (x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2), (x_3, y_3, z_3, w_3) be any three points, and λ, μ, ν the areal coordinates of any point in their plane referred to the triangle of which they are vertices; show that the equation to the section of any surface $\mathrm{U} = 0$ by the plane will be obtained by substituting for x, y, z, w from the equations

$$(\lambda + \mu + \nu)\,x = \lambda x_1 + \mu x_2 + \nu x_3,$$
$$(\lambda + \mu + \nu)\,y = \lambda y_1 + \mu y_2 + \nu y_3,$$
$$(\lambda + \mu + \nu)\,z = \lambda z_1 + \mu z_2 + \nu z_3,$$
$$(\lambda + \mu + \nu)\,w = \lambda w_1 + \mu w_2 + \nu w_3.$$

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8969. If the ternary n-ic be written

$$ax^n + \frac{n}{1}(b_1y + b_2z)x^{n-1} + \frac{n(n-1)}{1.2}(c_1y^2 + 2c_2y^2 + c_3z^2)x^{n-2} + \&c.,$$

and $ax + b_1y + b_2z$ be written for a,
$b_1x + c_1y + c_2z$ be written for b_1,
$b_2x + c_2y + c_3z$ be written for b_2, and so on,

in any invariant or covariant; the result will be a covariant of the $(n+1)$-ic

$$ax^{n+1} + \frac{n+1}{1}(b_1y + b_2x)x^n + \&c.$$

8970. If X, Y ... U denote the determinants

$$\left\| \begin{matrix} x_1, & y_1, & z_1, & w_1, & u_1 \\ x_2, & y_2, & z_2, & w_2, & u_2 \\ x_3, & y_3, & z_3, & w_3, & u_3 \\ x_4, & y_4, & z_4, & w_4, & u_4 \end{matrix} \right\|$$

and V_1, V_2, V_3, V_4 be the values of the quinary quadratic V when $(x_1, y_1, z_1, w_1, u_1)$, $(x_2, y_2, z_2, w_2, u_2)$, &c. are put for (x, y, z, w, u), and $S_{1.2}$, &c. stand for $\frac{1}{2}\left(x_1\frac{d}{dx_2} + y_1\frac{d}{dy_2} + \ldots\right)V_2$, &c.,

$$4\begin{vmatrix} V_1, & S_{1.2}, & S_{1.3}, & S_{1.4} \\ S_{1.2}, & V_2, & S_{2.3}, & S_{2.4} \\ S_{1.3}, & S_{2.3}, & V_3, & S_{3.4} \\ S_{1.4}, & S_{2.4}, & S_{3.4}, & V_4 \end{vmatrix} = AX^2 + BY^2 + \&c.,$$

where A, B, &c. are the first minors of the discriminant of V.

8971. If $(x_1, y_1, z_1, w_1, u_1)$, $(x_2, y_2, z_2, w_2, u_2)$, $(x_3, y_3 \ldots u_3)$, $(x_4, y_4 \ldots u_4)$, be four points, and λ, μ, ν, π the tetrahedral coordinates of any point in the linear hyper-surface determined by them referred to the tetrahedron of which they are the vertices; show that the equation to the surface common to this linear hyper-surface and the n-ic hyper-surface $U = 0$ will be obtained by substituting for x, y, z, w, u from the equations

$$\begin{aligned}
(\lambda+\mu+\nu+\pi)x &= \lambda x_1 + \mu x_2 + \nu x_3 + \pi x_4 \\
(\lambda+\mu+\nu+\pi)y &= \lambda y_1 + \mu y_2 + \nu y_3 + \pi y_4 \\
(\lambda+\mu+\nu+\pi)z &= \lambda z_1 + \mu z_2 + \nu z_3 + \pi z_4 \\
(\lambda+\mu+\nu+\pi)w &= \lambda w_1 + \mu w_2 + \nu w_3 + \pi w_4 \\
(\lambda+\mu+\nu+\pi)u &= \lambda u_1 + \mu u_2 + \nu u_3 + \pi u_4.
\end{aligned}$$

8972. The Hessian of a cubic curve is the envelope of the straight lines, two of the poles of which with respect to the cubic coincide.

8973. If $(n)^{(r)}$ denote $n(n-1)\ldots\ldots(n-r+1)$,

show that $$n^r = (n)^{(r)} + {}^rA_1(n)^{(r-1)} + {}^rA_2(n)^{(r-2)} + \&c.,$$

where $${}^rA_p = \frac{(r-p)^{r-1}}{(r-p-1)!} - \frac{(r-p-1)^{r-1}}{1!\,(r-p-2)!} + \frac{(r-p-2)^{r-1}}{2!\,(r-p-3)!} - \frac{(r-p-3)^{r-1}}{3!\,(r-p-4)!}$$
$$= \Delta^{r-p}0^r \div (r-p)! \qquad + \&c.$$

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8974. Show (1) $S\alpha\beta\gamma\delta = S.\alpha\delta\gamma\beta = S.\beta\alpha\delta\gamma =$ &c. ;

(2) $S.(\beta\gamma V\gamma\alpha\beta + \gamma\alpha V\alpha\beta\gamma + \alpha\beta V\beta\gamma\alpha) = -S.\alpha\beta\gamma(S\beta\gamma + S\gamma\alpha + S\alpha\beta)$;

(3) $\alpha V\beta\gamma + \beta V\gamma\alpha + \gamma V\alpha\beta = 3S\alpha\beta\gamma$.

8975. If a body acted on by no forces be set in motion by an impulsive couple G, then, ω_1, ω_2, ω_3 being the angular velocities about any rectangular axes through the fixed point and fixed in the body, and A, B, C, D, E, F the products and moments of inertia about them,

$$G^2 = (A^2 + E^2 + F^2)\,\omega_1^2 + (B^2 + F^2 + D^2)\,\omega_2^2 + (C^2 + D^2 + E^2)\,\omega_3^2$$
$$- 2\,(2AD + BD + CD + EF)\,\omega_2\omega_3 - 2\,(2BE + CE + AE + FD)\,\omega_3\omega_1$$
$$- 2\,(2CF + AF + BF + DE)\,\omega_1\omega_2.$$

8976. Show that, if (1 . 2), (2 . 3), &c. denote the edges of a simplicissimum in space of n dimensions (see Quest. 8242), the squared radius of the circumscribed hyper-sphere is

$$-\tfrac{1}{2}\begin{vmatrix} 0, & (1.2)^2, & (1.3)^2\ldots \\ (1-2)^2, & 0, & (2.3)^2\ldots \\ \ldots & \ldots & \ldots \\ (1.n+1)^2, & (2.n+1)^2, & (3.n+1)^2\ldots \end{vmatrix} \div \begin{vmatrix} 0, & 1, & 1, & 1 \ldots \\ 1, & 0, & (1.2)^2, & (1.3)^2\ldots \\ 1, & (1.2)^2, & 0, & (2.3)^2\ldots \\ \ldots & \ldots & \ldots & \ldots \\ 1, & (1.n+1)^2, & (2.n+1)^2, & (3.n+1)^2\ldots \end{vmatrix}$$

which reduces to $\dfrac{n}{2\,(n+1)}$ when each of the edges is unity.

Also, show that, if d be the distance of the mean centre of equal masses at the vertices of the simplicissimum from the centre of the hyper-sphere, the same quantity equals $d^2 + \dfrac{1}{(n+1)^2}\{(1.2)^2 + (2.3)^2 + (3.1)^2 + \&c.\}$.

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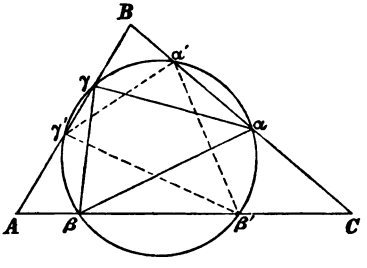
APPENDIX II.

CIRCULAR NOTES, BY R. TUCKER, M.A.



Let the circum-circle of the in-triangle $\alpha\beta\gamma$ of the triangle ABC cut the sides again in points α', β', γ'.* I propose to find some properties of these triangles and of their common circum-circle under conditions hereafter to be assigned.

Assume the angles

$$\left.\begin{aligned} A\gamma'\beta' = A\beta\gamma = \lambda_1,\ A\beta'\gamma' = A\gamma\beta = \lambda'_1 \\ B\alpha'\gamma' = B\gamma\alpha = \lambda_2,\ B\gamma'\alpha' = B\alpha\gamma = \lambda'_2 \\ C\beta'\alpha' = C\alpha\beta = \lambda_3,\ C\alpha'\beta' = C\beta\alpha = \lambda'_3 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(1),$$

and put

$$\left.\begin{aligned} B\alpha' = x',\ C\beta' = y',\ A\gamma' = z' \\ C\alpha = x,\ A\beta = y,\ B\gamma = z \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(2);$$

and, for shortness, write

$$p = \sin\lambda_1 / \sin\lambda'_1,\ q = \sin\lambda_2 / \sin\lambda'_2,\ r = \sin\lambda_3 / \sin\lambda'_3 \quad \ldots\ldots\ldots(3).$$

We have

$$\left.\begin{aligned} a = x' + y'r = x + zq \\ b = y' + z'p = y + xr \\ c = z' + x'q = z + yp \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(4).$$

From (4) we get

$$\left.\begin{aligned} x' = (a - br + cpr) / (1 + pqr) = 2\rho \sin\beta' \sin\lambda'_2 / \sin B \\ x = (a - bq + cqr) / (1 + pqr) = 2\rho \sin\gamma \sin\lambda'_3 / \sin C \\ a - x' = 2\rho \sin\gamma' \sin\lambda_3 / \sin C \\ a - x = 2\rho \sin\beta \sin\lambda_2 / \sin B \end{aligned}\right\} \quad \ldots\ldots\ldots(5).$$

* The figure is lettered in accordance with Fig. 1 of Mr. H. M. Taylor's Paper on the Intersections of a Circle and Triangle. (*Lond. Math. Soc. Proc.*, Vol. xv., p. 123.)

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Assuming the equation to the circum-circle to be

$$a\beta\gamma + b\gamma a + ca\beta = (a\alpha + b\beta + c\gamma)(\lambda\alpha + \mu\beta + \nu\gamma),$$

and bearing in mind that the points a, a', whose coordinates are

$$0,\ x\sin C,\ (a-x)\sin B\ ;\quad 0,\ (a-x')\sin C,\ x'\sin B,$$

are on the circle, we get, by substitution and subtraction, the equation

$$\mu c - \nu b = x' - x,$$

and also $$\nu a - \lambda c = y' - y, \quad \lambda b - \mu a = z' - z,$$

whence
$$\left.\begin{aligned} bc\lambda &= y(b-y') = t_a^2 \\ ca\mu &= t_b^2, \quad ab\nu = t_c^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

if t_a, t_b, t_c denote the lengths of the tangents drawn from the angular points of the triangle; whence the equation required may be written

$$abc(a\beta\gamma + \ldots + \ldots) = (a\alpha + \ldots + \ldots)(aat_a + b\beta t_b^2 + c\gamma t_c^2) \ldots\ldots\ldots\ldots(7),$$

showing that the radical axis of any of these circles and the circum-circle has for its equation

$$^{*}\ aat_a^2 + b\beta t_b^2 + c\gamma t_c^2 = 0.$$

If the polars of A, B, C, with respect to this circle, meet the opposite sides in L, M, N, we get that these points lie on the line

$$\frac{a\alpha}{t_b^2 + t_c^2 - a^2} + \frac{b\beta}{t_c^2 + t_a^2 - b^2} + \frac{c\gamma}{t_a^2 + t_b^2 - c^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots(8).$$

From equations (5) we obtain, if ρ is the radius of the circle,

$$\rho\left[\sin\lambda_1 \sin\lambda_2 \sin\lambda_3 + \sin\lambda'_1 \sin\lambda'_2 \sin\lambda'_3\right] = R\sin A \sin B \sin C \ldots(9).$$

We come now to the special cases considered in these notes.†

I. Assume the angles to be such that

$$\left.\begin{aligned} \lambda_1 &= nB - (n-1)C, & \lambda'_1 &= nC - (n-1)B \\ \lambda_2 &= nC - (n-1)A, & \lambda'_2 &= nA - (n-1)C \\ \lambda_3 &= nA - (n-1)B, & \lambda'_3 &= nB - (n-1)A \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(10).$$

From these values we see that

$$\left.\begin{aligned} \alpha &= n\pi - (3n-1)A, & \alpha' &= (3n-2)A - (n-1)\pi \\ \beta &= n\pi - (3n-1)B, & \beta' &= (3n-2)B - (n-1)\pi \\ \gamma &= n\pi - (3n-1)C, & \gamma' &= (3n-2)C - (n-1)\pi \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(11),$$

whence we get

$$\alpha + \alpha' = B + C, \quad \beta + \beta' = C + A, \quad \gamma + \gamma' = A + B.$$

Also
$$\begin{aligned} \alpha\gamma\alpha' &= (3n-2)C - (n-1)\pi - \left[nB - (n-1)A\right] \\ &= (3n-2)C - nB - (n-1)C - (n-1)B \\ &= (2n-1)(C-B) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12). \end{aligned}$$

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* It has been brought to my notice that a similar equation in tri-angular coordinates is given by Prof. Wolstenholme in his book of Problems (No. 1318.) The equation deserves a place in our text-books.

† The suppositions as to the angles were suggested to the writer in consequence of results arrived at in his Note on "The Sine-triple-Angle" circle, in the *Messenger of Mathematics*, December 1886; see also *Educational Times* for December.

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The following circles belong to this group :—

1° $n = \frac{1}{2}$, the in-circle.

2° $n = 1$ or 0, the nine-point circle (the two triangles are the medial and pedal triangles.)

3° $n = \frac{1}{3}$ or $\frac{2}{3}$, the circle about an inscribed equilateral triangle.

II. Assume the angles to be such that

$$\left.\begin{aligned} \lambda_1 &= \pi - n\mathrm{A}, & \lambda'_1 &= (n-1)\,\mathrm{A} \\ \lambda_2 &= \pi - n\mathrm{B}, & \lambda'_2 &= (n-1)\,\mathrm{B} \\ \lambda_3 &= \pi - n\mathrm{C}, & \lambda'_3 &= (n-1)\,\mathrm{C} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (13).$$

From these values we see that

$$\left.\begin{aligned} \alpha &= n\mathrm{C} - (n-1)\,\mathrm{B}, & \alpha' &= n\mathrm{B} - (n-1)\,\mathrm{C} \\ \beta &= n\mathrm{A} - (n-1)\,\mathrm{C}, & \beta' &= n\mathrm{C} - (n-1)\,\mathrm{A} \\ \gamma &= n\mathrm{B} - (n-1)\,\mathrm{A}, & \gamma' &= n\mathrm{A} - (n-1)\,\mathrm{B} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (14),$$

and $$\alpha\gamma\alpha' = n\mathrm{A} - (n-1)\,\mathrm{B} - (n-1)\,\mathrm{C} = (2n-1)\,\mathrm{A} - (n-1)\,\pi \quad \ldots\ldots (15).$$

When $n = 2$, this circle becomes the "sine-triple-angle" circle of the triangle, and for any integral value of n the intercepts on the sides are

$$\left.\begin{aligned} &\alpha\alpha' : \beta\beta' : \gamma\gamma' = \sin(2n-1)\,\mathrm{A} : \sin(2n-1)\,\mathrm{B} : \sin(2n-1)\,\mathrm{C} \\ &\text{[For a value of the form } (2k+1)/2\text{, the intercepts are as} \\ &\cos 2k\mathrm{A} : \cos 2k\mathrm{B} : \cos 2k\mathrm{C}\text{, and the cosine-circle is a particular} \\ &\text{case.] Again,} \\ &\beta\gamma' : \gamma\alpha' : \alpha\beta' = \sin(n-1)\,\mathrm{A} : \sin(n-1)\,\mathrm{B} : \sin(n-1)\,\mathrm{C} \\ &\gamma\beta' : \alpha\gamma' : \beta\alpha' = \sin n\mathrm{A} : \sin n\mathrm{B} : \sin n\mathrm{C} \end{aligned}\right\} \quad \ldots (16).$$

III. We may express the angles of the in-triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$, thus—

$$\left.\begin{aligned} \beta\alpha\gamma &= \mathrm{C} + \lambda'_3 - \lambda'_2 \\ \gamma\beta\alpha &= \mathrm{A} + \lambda'_1 - \lambda'_3 \\ \alpha\gamma\beta &= \mathrm{B} + \lambda'_2 - \lambda'_1 \\ \beta'\alpha'\gamma' &= \mathrm{B} + \lambda'_3 - \lambda'_3 \\ \gamma'\beta'\alpha' &= \mathrm{C} + \lambda'_3 - \lambda'_1 \\ \beta'\gamma'\alpha' &= \mathrm{A} + \lambda'_1 - \lambda'^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (17);$$

also

$$\left.\begin{aligned} \alpha\gamma\alpha' &= \mathrm{A} + 2\lambda'_1 - (\lambda'_1 + \lambda'_2 + \lambda'_3) = \mathrm{A} + 2\lambda'_1 - \Lambda \\ \beta\gamma\beta' &= \mathrm{B} + 2\lambda'_2 - \Lambda \\ \gamma\alpha\gamma' &= \mathrm{C} + 2\lambda'_3 - \Lambda \end{aligned}\right\} \quad \ldots\ldots (18).$$

If $\lambda'_1 = \lambda'_2 = \lambda'_3$, we have the "Tucker" circles.

If $\lambda'_1 = n(\frac{1}{2}\pi - \mathrm{A})$, $\lambda'_2 = n(\frac{1}{2}\pi - \mathrm{B})$, $\lambda'_3 = n(\frac{1}{2}\pi - \mathrm{C})$,

so that $\Lambda = n\pi/2$ and $\alpha\gamma\alpha' = n\frac{1}{2}\pi - (2n-1)\,\mathrm{A}$, we get the group of circles considered under II., and $n = 1$ gives the cosine-circle.

If $\lambda'_1 - \lambda'_3 = \frac{1}{3}\pi - \mathrm{A}$, $\lambda'_2 - \lambda'_2 = \frac{1}{3}\pi - \mathrm{B}$, $\lambda'_2 - \lambda'_1 = \frac{1}{3}\pi - \mathrm{C}$, with $\Lambda = k\pi$, we have $3\lambda'_1 = k\pi + \mathrm{B} - \mathrm{A}$, $3\lambda'_2 = k\pi + \mathrm{C} - \mathrm{B}$, $3\lambda'_3 = k\pi + \mathrm{A} - \mathrm{C}$;

in this case, if $k = 1$, we have the circum-circle of an in-equilateral triangle.

November, 1886.

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